v-Calculus on $\mathbb{C}^n$

Algebraic Structure, Integration, Constraints on the Field v, v-Taylor Series, v-Differential Equations, v-Elementary Functions, and Extension to Higher Dimensions

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Abstract

We develop a calculus on open subsets of $\mathbb{C}^n$ intermediate between real differentiability and complex analyticity. Given a smooth nonvanishing field $v : U \to \mathbb{C}^n \setminus \{0\}$, a function $\phi : U \to \mathbb{C}$ is called *v-differentiable* if $d\phi(v)(z) \in \mathbb{R}$ for all $z \in U$. The *v-derivative* $\partial_v\phi(z) := d\phi(v)(z) \in \mathbb{R}$ produces a real number from a complex function in any dimension. We establish: (i) algebraic structure of $\mathcal{D}_v(U)$ (real vector space, failure as $\mathbb{C}$-module, Leibniz rule, chain rule); (ii) a fundamental theorem of integration recovering only the real part, with imaginary part frozen as topological invariant Ω_γ; (iii) classification of fields v by integrability ($\omega_v = \sum v_k dz_k$ closed), normalization ($\omega_v(v) = 1$), and residue structure; (iv) v-Taylor series with real coefficients and explicit radius of convergence $R_v = \operatorname{dist}(\Phi(z_0), \operatorname{Sing}(F))$, in which Ω_γ is provably invisible to every order; (v) v-differential equations reducing via $w = \Phi(z)$ to real first-order ODEs parametrized by Ω_γ, unifying the A_n hierarchy as solutions of $\partial_{v_n}\phi = -1$; (vi) a complete family of *v-elementary functions* — v-exponential $\mathbf{E}_v = e^{E_1(\lambda z)}$ satisfying $\partial_v \mathbf{E}_v = \mathbf{E}_v$, v-trigonometric functions, v-powers, and v-logarithm, all generated by the principle $F \circ \Phi$ with F classical elementary, and carrying Ω_γ intrinsically in their phase; (vii) a *v-linearization theorem for integration* showing that classically intractable integrals of the form $\int g(E_1(\lambda z))/(ze^{\lambda z})\, dz$ reduce exactly to $G(E_1(\lambda z_1)) - G(E_1(\lambda z_0))$, where $G' = g$; (viii) a complete *v-Fourier theory* — orthogonal basis $\{\mathbf{S}_{v,n}, \mathbf{C}_{v,n}\}$, v-Fourier series with real coefficients, v-Parseval identity, and v-Fourier transform, all with numerical verification at machine precision; and (ix) extension to $\mathbb{C}^n$, revealing a dimensional reduction theorem: the v-calculus in dimension n is always isomorphic to a one-dimensional calculus in the coordinate Φ, with the genuine n-dimensional novelty concentrated in the topology of the fibres $\{\Phi(z) = c\}$. The *maximal bifurcation theorem* — that Ω_γ is invisible to every operation of the v-calculus and determined solely by the initial condition — holds in all dimensions. The real calculus and complex analysis are recovered as special cases, establishing the v-calculus as a strictly more general framework.

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1 Motivation and Context

1.1 The starting observation

Consider an analytic function $\phi : U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$, and a smooth curve $\gamma : [0, T) \to U$. The derivative of ϕ along γ at time t is the complex number

$$\frac{d}{dt}\phi(\gamma(t)) = \phi'(\gamma(t)) \cdot \dot{\gamma}(t) \in \mathbb{C}.$$

In general this is complex. The natural question is: for which functions ϕ and which curves γ is this derivative *real*?

A function satisfying $\phi'(\gamma(t)) \cdot \dot{\gamma}(t) \in \mathbb{R}$ for all t will be called γ-*real*. One can show directly that:

(i) ϕ is γ-real if and only if $\phi'(z)$ is a real multiple of $1/\dot{\gamma}(\gamma^{-1}(z))$ at each point z on γ.

(ii) The imaginary part $\mathrm{Im}(\phi(\gamma(t)))$ is constant along γ — a frozen topological invariant Ω_γ.

(iii) For the A_n hierarchy $\dot{u} = u^n e^{\lambda u}$, the unique γ-real function along the solution curve is $\phi_n = \lambda^{n-1} E_n(\lambda \cdot)$.

This curve-dependent framework has a fundamental limitation: the condition is tied to a specific curve γ, not to the domain U itself.

1.2 The central idea

The v-calculus resolves this limitation by replacing the curve γ with a *field* $v : U \to \mathbb{C}\setminus\{0\}$ defined at every point. Instead of requiring $\phi'(\gamma(t)) \cdot \dot{\gamma}(t) \in \mathbb{R}$ along a fixed curve, we require

$$\phi'(z) \cdot v(z) \in \mathbb{R} \quad \text{for all } z \in U.$$

This is a pointwise condition, intrinsic to U, independent of any particular trajectory. The field v encodes the direction of differentiation at each point and becomes the primitive object of the calculus.

The curve-dependent setting is recovered as the special case $v(z) = \dot{\gamma}(\gamma^{-1}(z))$ — the velocity field of γ. But v is now free, and the calculus is defined globally on U.

2 Algebraic Structure of $\mathcal{D}_v(U)$

Let $U \subset \mathbb{C}$ be an open set and $v : U \to \mathbb{C} \setminus \{0\}$ a smooth nonvanishing field.

Definition 1 (v-differentiable function). *An analytic function $\phi : U \to \mathbb{C}$ is called* v-differentiable *if*

$$\phi'(z) \cdot v(z) \in \mathbb{R} \quad \textit{for all } z \in U.$$

The v-derivative *of ϕ at z is*

$$\partial_v \phi(z) := \phi'(z) \cdot v(z) \in \mathbb{R}.$$

We denote by $\mathcal{D}_v(U)$ the set of all v-differentiable functions on U.

Remark 1. *The condition $\phi'(z)\cdot v(z) \in \mathbb{R}$ is equivalent to $\arg(\phi'(z)) \equiv \arg(v(z)) \pmod{\pi}$: the derivative $\phi'(z)$ must be a real multiple of $1/v(z)$. This is a single real constraint at each point, strictly weaker than the Cauchy-Riemann conditions (which require the derivative to be direction-independent).*

2.1 Vector space structure

Proposition 1 (Real vector space). *$\mathcal{D}_v(U)$ is a vector space over $\mathbb{R}$.*

Proof. If $\phi, \psi \in \mathcal{D}_v(U)$ and $\alpha, \beta \in \mathbb{R}$, then

$$(\alpha\phi + \beta\psi)'(z) \cdot v(z) = \alpha(\phi'(z) \cdot v(z)) + \beta(\psi'(z) \cdot v(z)) \in \mathbb{R}. \qquad \square$$

Proposition 2 (Failure as $\mathbb{C}$-module). *$\mathcal{D}_v(U)$ is* not *a vector space over $\mathbb{C}$. Specifically, if $\phi \in \mathcal{D}_v(U)$ is non-constant, then $i\phi \notin \mathcal{D}_v(U)$.*

Proof. $(i\phi)'(z) \cdot v(z) = i \cdot (\phi'(z) \cdot v(z)) \in i\mathbb{R}$, which is purely imaginary and therefore not real (unless zero). Multiplying by i maps $\mathbb{R}$ to $i\mathbb{R}$, exiting the space. $\square$

Remark 2. *This failure is not a defect but a structural feature. It reflects that the v-calculus is genuinely intermediate between real and complex: it retains analyticity of ϕ on U while imposing a real-valued directional constraint. The "scalar field" of this calculus is $\mathbb{R}$, not $\mathbb{C}$.*

2.2 Leibniz rule and the product constraint

Proposition 3 (Leibniz rule with constraint). *Let $\phi, \psi \in \mathcal{D}_v(U)$. Then $\phi\psi \in \mathcal{D}_v(U)$ if and only if*

$$\partial_v\phi(z) \cdot \mathrm{Im}(\psi(z)) + \partial_v\psi(z) \cdot \mathrm{Im}(\phi(z)) = 0 \quad \text{for all } z \in U. \tag{1}$$

When this holds:

$$\partial_v(\phi\psi)(z) = \partial_v\phi(z) \cdot \mathrm{Re}(\psi(z)) + \partial_v\psi(z) \cdot \mathrm{Re}(\phi(z)).$$

Proof. By the product rule,

$$(\phi\psi)'(z) \cdot v(z) = \psi(z) \cdot (\phi'(z)v(z)) + \phi(z) \cdot (\psi'(z)v(z)) = \psi(z) \cdot \partial_v\phi(z) + \phi(z) \cdot \partial_v\psi(z).$$

Since $\partial_v\phi(z), \partial_v\psi(z) \in \mathbb{R}$, this is real if and only if

$$\mathrm{Im}(\psi(z) \cdot \partial_v\phi(z) + \phi(z) \cdot \partial_v\psi(z)) = \partial_v\phi(z) \cdot \mathrm{Im}(\psi(z)) + \partial_v\psi(z) \cdot \mathrm{Im}(\phi(z)) = 0. \qquad \square$$

Corollary 1 (Real-valued subfunctions form a ring). *The subspace $\mathcal{D}_v(U) \cap C^\infty(U, \mathbb{R})$ of real-valued v-differentiable functions is a ring under pointwise multiplication.*

Proof. If ϕ, ψ are real-valued, $\mathrm{Im}(\phi) = \mathrm{Im}(\psi) = 0$, so condition (1) is automatically satisfied. $\square$

2.3 Chain rule

Proposition 4 (Chain rule). *Let $\phi \in \mathcal{D}_v(U)$ and let $F : V \to \mathbb{C}$ be analytic on an open set $V \supset \phi(U)$, with F' real-valued on $\phi(U)$. Then $F \circ \phi \in \mathcal{D}_v(U)$ and*

$$\partial_v(F \circ \phi)(z) = F'(\phi(z)) \cdot \partial_v \phi(z).$$

Proof. $(F \circ \phi)'(z) \cdot v(z) = F'(\phi(z)) \cdot \phi'(z) \cdot v(z) = F'(\phi(z)) \cdot \partial_v \phi(z) \in \mathbb{R}$, since $F'(\phi(z)) \in \mathbb{R}$ and $\partial_v \phi(z) \in \mathbb{R}$. □

Remark 3. *The condition that F' be real-valued on $\phi(U)$ is the natural analogue of the Cauchy-Riemann condition for compositions in this calculus. Functions F with real derivative on an arc are exactly those expressible as $F(w) = G(\mathrm{Re}(w))$ locally, where $G : \mathbb{R} \to \mathbb{C}$ is smooth.*

3 Integration

3.1 The v-integral

Definition 2 (v-integral). *Let $\gamma : [0, T] \to U$ be a smooth curve and $f : U \to \mathbb{R}$ smooth. The* v-integral *of f along γ is*

$$\int_\gamma f \, d_v t := \int_0^T f(\gamma(t)) \, dt.$$

Remark 4. *This is not a contour integral in the classical sense. It integrates a real-valued function along the real parameter t of γ, without reference to the complex structure of $\mathbb{C}$. The complex geometry enters only through γ itself.*

3.2 Fundamental theorem of v-calculus

Theorem 1 (Fundamental theorem). *Let $\phi \in \mathcal{D}_v(U)$ and $\gamma : [0, T] \to U$ a smooth curve. Then:*

$$\int_\gamma \partial_v \phi \, d_v t = \mathrm{Re}(\phi(\gamma(T))) - \mathrm{Re}(\phi(\gamma(0))). \tag{2}$$

Proof. Since $\partial_v \phi(\gamma(t)) = \phi'(\gamma(t)) \cdot \dot\gamma(t) \in \mathbb{R}$,

$$\int_0^T \partial_v \phi(\gamma(t)) \, dt = \int_0^T \mathrm{Re}(\phi'(\gamma(t)) \cdot \dot\gamma(t)) \, dt = \mathrm{Re}\left(\int_0^T \frac{d}{dt} \phi(\gamma(t)) \, dt \right) = \mathrm{Re}(\phi(\gamma(T))) - \mathrm{Re}(\phi(\gamma(0))).$$

□

3.3 The fundamental decomposition

Corollary 2 (Fundamental decomposition). *For any $\phi \in \mathcal{D}_v(U)$ and smooth curve $\gamma : [0, T] \to U$,*

$$\phi(\gamma(t)) = \underbrace{\mathrm{Re}(\phi(\gamma(0))) + \int_0^t \partial_v \phi \, d_v s}_{\text{dynamic part — evolves at rate } \partial_v \phi} + i \underbrace{\Omega_\gamma}_{\text{topological part — frozen}}. \tag{3}$$

The imaginary part $\Omega_\gamma = \mathrm{Im}(\phi(\gamma(t)))$ is constant along γ (up to branch-cut corrections $2\pi\mathcal{K}(t)$).

Proof. Taking imaginary parts in $\phi(\gamma(t)) - \phi(\gamma(0)) \in \mathbb{R}$ (from Theorem 1 with $s = 0$) gives $\mathrm{Im}(\phi(\gamma(t))) = \mathrm{Im}(\phi(\gamma(0)))$. □

Remark 5 (Two species of information). *The decomposition (3) reveals that every v-differentiable function carries two fundamentally different kinds of information simultaneously:*

- *The* dynamic part $\mathrm{Re}(\phi)$*, accessible to the v-derivative and integration, evolving along γ.*
- *The* topological part Ω_γ*, invisible to ∂_v, frozen along γ, and determined entirely by the initial condition.*

There is no analogue of this decomposition in the real calculus (which has no imaginary part) or in complex analysis (where the imaginary part is not conserved along curves). It is a genuinely new phenomenon.

3.4 v-primitives and the path integral

Definition 3 (v-primitive). *A function $\Phi \in \mathcal{D}_v(U)$ is a* v-primitive *of $f : U \to \mathbb{R}$ if $\partial_v \Phi(z) = f(z)$ for all $z \in U$.*

Proposition 5 (Existence of v-primitives). *A smooth function $f : U \to \mathbb{R}$ admits a v-primitive if and only if f/v admits an analytic primitive on U, i.e., there exists analytic $\Phi : U \to \mathbb{C}$ with $\Phi'(z) = f(z)/v(z)$.*

Proof. $\partial_v \Phi = f$ requires $\Phi'(z) \cdot v(z) = f(z)$, hence $\Phi'(z) = f(z)/v(z)$. Such Φ exists analytically on U iff f/v has an analytic primitive, which on a simply connected domain is always the case for analytic f/v. □

Definition 4 (v-path integral). *For $f : U \to \mathbb{R}$ and points $z_0, z_1 \in U$, define*

$$\int_{z_0}^{z_1} f \, d_v z := \int_0^T f(\gamma(t)) \, dt,$$

where γ is any integral curve of v with $\gamma(0) = z_0$, $\gamma(T) = z_1$.

Proposition 6 (Path independence). *If U is simply connected, the v-path integral is independent of the choice of integral curve of v.*

Proof. Two integral curves of v through z_0 coincide by uniqueness of solutions to $\dot{\gamma} = v(\gamma)$. □

Remark 6 (Absence of a Stokes theorem). *Unlike the classical line integral, the v-path integral has no natural Stokes theorem: there is no notion of "boundary" in this calculus, because all integration follows the unidirectional flow of v. Topology enters not through integration over cycles but through the frozen invariant Ω_γ. This is a fundamental structural difference from both real and complex calculi.*

4 Constraints on the Field v

4.1 Integrability

Definition 5 (Integrable field). *A field $v : U \to \mathbb{C} \setminus \{0\}$ is* integrable *if $1/v$ admits an analytic primitive on U: there exists analytic $\Phi : U \to \mathbb{C}$ with*

$$\Phi'(z) = \frac{1}{v(z)}.$$

The function Φ is called the generating primitive *of v.*

Proposition 7 (Richness under integrability). *If v is integrable with generating primitive Φ, then for any analytic function $F : \mathbb{C} \to \mathbb{C}$ with F' real-valued, one has $F \circ \Phi \in \mathcal{D}_v(U)$ and*

$$\partial_v(F \circ \Phi)(z) = F'(\Phi(z)) \in \mathbb{R}.$$

In particular, $\mathcal{D}_v(U)$ contains a family of functions parametrized by all such F.

Proof. $(F \circ \Phi)'(z) \cdot v(z) = F'(\Phi(z)) \cdot \Phi'(z) \cdot v(z) = F'(\Phi(z)) \cdot (1/v(z)) \cdot v(z) = F'(\Phi(z)) \in \mathbb{R}$. □

Remark 7. *The generating primitive Φ is the* master object *of the calculus: all v-differentiable functions are, in a precise sense, functions of Φ.*

4.2 The obstruction: residues generate topology

On a non-simply-connected domain, Φ may be multivalued. The obstruction to global integrability is measured by residues.

Proposition 8 (Residue obstruction). *Let $U = \mathbb{C} \setminus \{z_1, \ldots, z_k\}$ and v analytic on U. The generating primitive Φ is single-valued on U if and only if*

$$\mathrm{Res}_{z=z_j} \frac{1}{v(z)} = 0 \quad \textit{for all } j = 1, \ldots, k.$$

When a residue is non-zero, Φ acquires a branch cut, and the jump of $\mathrm{Im}(\Phi)$ across this cut is $2\pi \cdot \mathrm{Res}(1/v, z_j) \in 2\pi\mathbb{Z}$.

Proof. Standard: $\oint_\gamma dz/v(z) = 2\pi i \sum_j \mathrm{Res}(1/v, z_j)$ by the residue theorem. This vanishes iff all residues are zero, iff Φ is single-valued. □

Corollary 3 (Residues force topological invariants). *The following implications hold:*

$$\mathrm{Res}(1/v, z_j) \neq 0 \implies \Phi \textit{ multivalued} \implies \Omega_\gamma \in 2\pi\mathbb{Z}\textit{-torsor}.$$

Non-zero residues are precisely the generators of topological invariants in the v-calculus.

4.3 Completeness

Definition 6 (Complete field). *An integrable field v is* complete *on U if its generating primitive $\Phi : U \to \Phi(U)$ is a biholomorphism (injective, analytic, with analytic inverse).*

Proposition 9 (Characterization of complete fields). *v is complete if and only if Φ is injective on U, in which case $\mathcal{D}_v(U)$ separates points of U and the v-calculus on U is isomorphic to the real calculus on $\Phi(U) \subset \mathbb{C}$ via the coordinate change $w = \Phi(z)$.*

Proof. If Φ is injective, then $\mathrm{Re}(\Phi(z))$ separates points ($z \neq w$ implies $\Phi(z) \neq \Phi(w)$, hence $\mathrm{Re}(\Phi(z)) \neq \mathrm{Re}(\Phi(w))$ or $\mathrm{Im}(\Phi(z)) \neq \mathrm{Im}(\Phi(w))$, and in either case some element of $\mathcal{D}_v(U)$ distinguishes them). The isomorphism $\phi \mapsto \phi \circ \Phi^{-1}$ identifies v-derivatives on U with ordinary real derivatives on $\Phi(U)$. □

Remark 8 (Canonical coordinate change). *If v is complete with generating primitive Φ, then*

$$\partial_v \phi(z) = \frac{d}{dw}(\phi \circ \Phi^{-1})(w)\Big|_{w=\Phi(z)}.$$

Every complete v-calculus is locally equivalent to the real calculus, via the coordinate Φ. The geometric richness of the v-calculus lies entirely in how Φ distorts U inside $\mathbb{C}$.

4.4 Classification summary

The following table organizes the three levels of richness of the v-calculus according to the properties of v.

Type of v	$\mathcal{D}_v(U)$	Invariant Ω_γ	Canonical example
Non-integrable	trivial (constants only)	none	$v(z) = \bar{z}$
Integrable, complete	rich, isomorphic to $\mathbb{R}$-calculus	trivial	$v \equiv 1$, $\Phi = z$
Integrable, with residues	rich, genuinely complex	$\in 2\pi\mathbb{Z}$-torsor	$v = ze^{\lambda z}$, $\Phi = E_1(\lambda z)$

Remark 9 (Three levels, one new). *The first level is degenerate. The second level is essentially the real calculus in disguise. Only the third level is genuinely new: it carries topological information invisible from real dynamics, generated by the residue structure of $1/v$. The $\mathcal{F}_\lambda$ class of [1] lives at this third level, with $v(z) = ze^{\lambda z}$, residue 1 at $z = 0$, and $\Phi = E_1(\lambda z)$.*

5 Special Cases and Connections

5.1 The curve-dependent setting as a special case

Proposition 10 (Recovery of curve-dependent calculus). *Let $\gamma : [0, T) \to U$ be a smooth curve and set $v(z) = \dot{\gamma}(\gamma^{-1}(z))$ at each point $z \in \gamma([0, T))$. Then a function ϕ is v-differentiable if and only if $\phi'(\gamma(t)) \cdot \dot{\gamma}(t) \in \mathbb{R}$ for all t, and $\partial_v \phi(\gamma(t)) = \frac{d}{dt}\mathrm{Re}(\phi(\gamma(t)))$. The topological invariant $\Omega_\gamma = \mathrm{Im}(\phi(\gamma(t)))$ of the v-calculus coincides with the frozen imaginary part along γ.*

Proof. Direct substitution of $v(z) = \dot{\gamma}(\gamma^{-1}(z))$ into Definition 1 gives $\phi'(\gamma(t)) \cdot \dot{\gamma}(t) \in \mathbb{R}$, and $\partial_v \phi(\gamma(t)) = \phi'(\gamma(t)) \cdot \dot{\gamma}(t) = \frac{d}{dt}\mathrm{Re}(\phi(\gamma(t)))$ by the fundamental theorem. □

Remark 10. *The curve-dependent setting is thus a special case of the v-calculus, obtained by restricting v to the velocity field of a fixed curve. The v-calculus is the intrinsic, coordinate-free version of this construction.*

5.2 Recovery of $\mathcal{F}_\lambda$ invariants

For the class $\mathcal{F}_\lambda : \dot{u} = ue^{\lambda u}$ with $\lambda = i\mu$, the velocity field is $v(z) = ze^{i\mu z}$, the generating primitive is $\Phi(z) = E_1(i\mu z)$, and the topological invariant is

$$\Omega_\gamma = \mathrm{Im}(E_1(i\mu u_0)) = \frac{\pi}{2} - \mathrm{Si}(\mu u_0),$$

recovering the exact first integral of [1] as a direct consequence of Corollary 3 and the fundamental decomposition (3).

5.3 The A_n hierarchy

For $A_n : \dot{u} = u^n e^{\lambda u}$, the velocity field is $v_n(z) = z^n e^{\lambda z}$ and the generating primitive is $\Phi_n(z) = \lambda^{n-1} E_n(\lambda z)$, with residue structure encoding the topological invariants $\Omega^{(n)}$ of [1], Section 8.

6 v-Taylor Series

6.1 The iteration problem

In the real calculus, $f'(x) \in \mathbb{R}$ and iteration is immediate: $f^{(n)}(x) = (f^{(n-1)})'(x)$. In the v-calculus, $\partial_v \phi(z) \in \mathbb{R}$ along integral curves of v, but a subtlety arises for iteration: a function analytic on an open set $U \subset \mathbb{C}$ that is real-valued on all of U must be constant (by the Cauchy-Riemann equations). So $\partial_v \phi$ cannot be both analytic on U and globally real-valued unless it is constant.

The resolution is that $\partial_v \phi(z) = \phi'(z) \cdot v(z)$ is a function $U \to \mathbb{C}$ whose restriction to integral curves of v is real, not a globally real function. Iteration is defined by applying ∂_v to this complex-valued function:

$$\partial_v^2 \phi(z) := \partial_v(\partial_v \phi)(z) = (\phi'(z)v(z))' \cdot v(z) = (\phi''(z)v(z) + \phi'(z)v'(z)) \cdot v(z).$$

6.2 Iterated v-derivatives via the generating primitive

The coordinate $w = \Phi(z)$ diagonalizes the iteration completely.

Theorem 2 (Iterated v-derivatives). *Let v be a complete integrable field on U with generating primitive Φ, and let $\phi \in \mathcal{D}_v(U)$. Define $F := \phi \circ \Phi^{-1}$, so that $\phi(z) = F(\Phi(z))$. Then for all $n \geq 1$:*

$$\partial_v^n \phi(z) = F^{(n)}(\Phi(z)). \tag{4}$$

Proof. By induction. For $n = 1$:

$$\partial_v \phi(z) = \phi'(z) \cdot v(z) = F'(\Phi(z)) \cdot \Phi'(z) \cdot v(z) = F'(\Phi(z)) \cdot \frac{1}{v(z)} \cdot v(z) = F'(\Phi(z)).$$

For the inductive step, suppose $\partial_v^{n-1}\phi(z) = F^{(n-1)}(\Phi(z))$. Then:

$$\partial_v^n\phi(z) = \partial_v(F^{(n-1)}\circ\Phi)(z) = (F^{(n-1)})'(\Phi(z))\cdot\Phi'(z)\cdot v(z) = F^{(n)}(\Phi(z))\cdot\frac{1}{v(z)}\cdot v(z) = F^{(n)}(\Phi(z)).$$

□

Corollary 4 (Reality of iterated derivatives). *If $F^{(n)}$ is real-valued on $\Phi(U)$, then $\partial_v^n\phi$ is real along every integral curve of v in U.*

6.3 The v-Taylor expansion

Definition 7 (v-Taylor series). *Let v be complete with generating primitive Φ, $\phi \in \mathcal{D}_v(U)$, and $z_0 \in U$. The* v-Taylor series *of ϕ at z_0 is:*

$$\phi(z) = i\Omega_\gamma + \sum_{n=0}^{\infty} \frac{\partial_v^n\phi(z_0)}{n!}\,(\Phi(z) - \Phi(z_0))^n, \tag{5}$$

where $\partial_v^0\phi(z_0) := \mathrm{Re}(\phi(z_0))$ and $\Omega_\gamma = \mathrm{Im}(\phi(z_0))$.

Theorem 3 (v-Taylor convergence). *The v-Taylor series (5) converges if and only if $F = \phi\circ\Phi^{-1}$ is analytic in a neighborhood of $w_0 = \Phi(z_0)$ in $\mathbb{C}$. Its radius of convergence in the w-coordinate is the distance from w_0 to the nearest singularity of F, and all coefficients $\partial_v^n\phi(z_0)/n! \in \mathbb{R}$ are real.*

Proof. By Theorem 2, the v-Taylor series is the ordinary Taylor series of F at w_0, transported to z-coordinates via Φ:

$$\sum_{n=0}^{\infty} \frac{F^{(n)}(w_0)}{n!}(w - w_0)^n = \sum_{n=0}^{\infty} \frac{\partial_v^n\phi(z_0)}{n!}(\Phi(z) - \Phi(z_0))^n.$$

Convergence, radius, and the reality of coefficients all follow from the corresponding properties of the Taylor series of F. □

6.4 The fundamental separation theorem

Theorem 4 (Topological invariant is Taylor-invisible). *Let $\phi \in \mathcal{D}_v(U)$ and $z_0 \in U$. The topological invariant $\Omega_\gamma = \mathrm{Im}(\phi(z_0))$ satisfies:*

$$\partial_v^n\phi(z_0) = 0 \quad \textit{for all } n \geq 1 \iff F = \phi\circ\Phi^{-1} \textit{ is constant on } \Phi(U),$$

in which case $\phi \equiv \mathrm{Re}(\phi(z_0)) + i\Omega_\gamma$ is a pure topological constant, invisible to all derivatives. More generally, Ω_γ does not appear in any coefficient of the v-Taylor series: it is the additive constant $i\Omega_\gamma$ that no differentiation can produce or detect.

Proof. The coefficients $\partial_v^n\phi(z_0)/n! = F^{(n)}(w_0)/n!$ are the real parts of the Taylor coefficients of F. Since $\Omega_\gamma = \mathrm{Im}(\phi(z_0)) = \mathrm{Im}(F(w_0))$, it contributes only to the imaginary part of the constant term $F(w_0)$, never to any derivative $F^{(n)}$ for $n \geq 1$. Hence Ω_γ is irrecoverable from the v-Taylor series alone. □

Remark 11 (Irrecoverability of Ω_γ). *Theorem 4 gives the sharpest possible statement: the topological invariant is not merely "hard to compute" from the Taylor expansion — it is* structurally absent *from the entire expansion. No finite or infinite collection of v-derivatives at any point can reconstruct Ω_γ. It must be supplied as initial data, exactly as a constant of integration in the real calculus — except that here the "constant" is complex and carries topological meaning.*

6.5 Three observations on the structure of v-Taylor series

Observation 1: Real coefficients in complex coordinates. The v-Taylor series is a power series in $(\Phi(z) - \Phi(z_0)) \in \mathbb{C}$ with all coefficients in $\mathbb{R}$. It is neither a real Taylor series (the variable $\Phi(z)$ is complex) nor a complex Taylor series (the coefficients are real). It is a new species: a real-coefficient expansion in a complex variable defined by the geometry of Φ.

Observation 2: Local triviality, global novelty. Locally, the v-calculus is isomorphic to the real calculus in the coordinate $w = \Phi(z)$. The v-Taylor series confirms this: it is exactly the real Taylor series of F transported by Φ. The genuine novelty of the v-calculus is therefore not local but global, concentrated in two places: the singularities of Φ (where $v = 0$ or residues are non-zero) and the global geometry of $\Phi(U)$.

Observation 3: Two irreconcilable parts. Every $\phi \in \mathcal{D}_v(U)$ decomposes as:

$$\phi = \underbrace{F \circ \Phi}_{\text{dynamic: captured by } v\text{-Taylor}} + \underbrace{i\Omega_\gamma}_{\text{topological: invisible to all derivatives}}.$$

These two parts are irreconcilable: no operation in the v-calculus converts one into the other. The v-calculus is therefore not just a calculus but a *bifurcated structure* — two parallel realities coexisting in a single function.

6.6 Radius of convergence of the v-Taylor series

Theorem 5 (Radius of convergence). *Let $\phi = F \circ \Phi \in \mathcal{D}_v(U)$ and $z_0 \in U$. The v-Taylor series of ϕ at z_0 converges in the w-coordinate on the disk*

$$\{w \in \mathbb{C} : |w - w_0| < R_v(\phi, z_0)\}, \quad w_0 = \Phi(z_0),$$

where

$$R_v(\phi, z_0) := \operatorname{dist}_{\mathbb{C}}(w_0, \operatorname{Sing}(F)) = \inf_{w^* \in \operatorname{Sing}(F)} |w_0 - w^*|. \tag{6}$$

In the original coordinate z, the series converges on the largest Φ-connected open subset of U containing z_0 on which F is analytic, i.e., on $\Phi^{-1}(\{|w - w_0| < R_v\}) \cap U$.

Proof. By Theorem 3, the v-Taylor series of ϕ at z_0 is the ordinary Taylor series of F at w_0, transported via Φ. By the Cauchy-Hadamard theorem, the Taylor series of F at w_0 converges exactly on the open disk of radius $R_v = \operatorname{dist}(w_0, \operatorname{Sing}(F))$ in the w-plane. The preimage under Φ of this disk gives the convergence domain in the z-coordinate. □

Remark 12 (Two distinct sources of singularity). *In the v-calculus, singularities arise from two independent sources:*

- Singularities of F*: these determine R_v and are intrinsic to the chosen v-elementary function.*
- Singularities of Φ*: these are the zeros of v and the branch points of the generating primitive; they determine the domain U on which the v-calculus is defined. They are independent of F and of the Taylor series convergence.*

The radius R_v measures only the first type. The second type constrains the domain but not the series.

Corollary 5 (Canonical radii). *For the canonical v-elementary functions with $v(z) = ze^{\lambda z}$ and $w_0 = E_1(\lambda z_0)$:*

Function ϕ	$F(w)$	$\mathrm{Sing}(F)$	$R_v(\phi, z_0)$
v-exponential $\mathbf{E}_v$	e^w	$\emptyset$	$+\infty$
v-sine $\mathbf{S}_v$	$\sin w$	$\emptyset$	$+\infty$
v-cosine $\mathbf{C}_v$	$\cos w$	$\emptyset$	$+\infty$
v-power $\mathbf{P}_v^n$	w^n	$\emptyset$	$+\infty$
v-logarithm $\mathbf{L}_v$	$\ln w$	$\{0\}$	$\lvert E_1(\lambda z_0)\rvert$
Bernoulli solution	$1/(C-w)$	$\{C\}$	$\lvert E_1(\lambda z_0) - C\rvert$

For v-elementary functions arising from entire functions F (exponential, trigonometric, powers), the v-Taylor series converges on all of $\Phi^{-1}(\mathbb{C}) = U$. The only source of finite radius is a singularity of F, localized in the w-plane at $w^* = \Phi(z^*)$, i.e., at the point z^* satisfying $E_1(\lambda z^*) = w^*$.

Remark 13 (Comparison with classical analysis). *In the real calculus, the radius of convergence of the Taylor series of f at x_0 is $\mathrm{dist}(x_0, \mathrm{Sing}(f))$ in $\mathbb{R}$ (or $\mathbb{C}$ for the complex extension). In the v-calculus, the formula is identical but lives in the Φ-coordinate: the geometry of convergence is entirely encoded by Φ. This is the quantitative expression of the local isomorphism between the v-calculus and the real calculus established in Proposition 9.*

7 v-Differential Equations

7.1 Definition and the reduction theorem

Definition 8 (v-differential equation). *A* v-differential equation *(v-ODE) is an equation of the form*

$$\partial_v \phi(z) = F(\phi(z), z), \tag{7}$$

where $F : \mathbb{C} \times U \to \mathbb{R}$ is a given smooth function. The constraint $F \in \mathbb{R}$ is necessary: since $\partial_v \phi \in \mathbb{R}$ by definition, the right-hand side must be real-valued.

The key tool is the coordinate $w = \Phi(z)$, which diagonalizes the equation.

Theorem 6 (Reduction theorem). *Every v-ODE (7) reduces, via $w = \Phi(z)$ and $G = \phi \circ \Phi^{-1}$, to the real first-order ODE*

$$G'(w) = F(G(w),\, \Phi^{-1}(w)), \tag{8}$$

in which $\Omega_\gamma = \mathrm{Im}(\phi(z_0))$ appears as a free real parameter in the initial condition $G(w_0) = \mathrm{Re}(\phi(z_0)) + i\Omega_\gamma$, but is not constrained by the equation itself.

Proof. By Theorem 2, $\partial_v \phi(z) = G'(\Phi(z))$. Substituting into (7): $G'(\Phi(z)) = F(G(\Phi(z)), z)$. Setting $w = \Phi(z)$ and $z = \Phi^{-1}(w)$ gives (8). The imaginary part of $G(w) = P(\Phi^{-1}(w)) + i\Omega_\gamma$ is constant (by the fundamental decomposition), so F acts only on $\mathrm{Re}(G(w)) = P(\Phi^{-1}(w))$, leaving Ω_γ free. □

Corollary 6 (Classification via real ODEs). *The classification of v-ODEs is isomorphic to the classification of real first-order ODEs, with one additional datum: the free parameter $\Omega_\gamma \in \mathbb{R}$. Every v-ODE is a one-parameter family of real ODEs, indexed by the topological invariant.*

7.2 The four canonical classes

Class 1 — Constant equation $\partial_v \phi = c$

Proposition 11 (Constant v-ODE). *The v-ODE $\partial_v \phi = c$, $c \in \mathbb{R}$, has unique solution*

$$\phi(z) = c\,\Phi(z) + a + i\Omega_\gamma, \quad a \in \mathbb{R}, \tag{9}$$

with $a = \mathrm{Re}(\phi(z_0)) - c\,\mathrm{Re}(\Phi(z_0))$ and $\Omega_\gamma = \mathrm{Im}(\phi(z_0))$ determined by the initial condition.

Proof. The reduced equation is $G'(w) = c$, giving $G(w) = cw + a$, $a \in \mathbb{R}$. Recomposing: $\phi(z) = G(\Phi(z)) = c\Phi(z) + a + i\Omega_\gamma$. □

Example 1 (The $\mathcal{F}_\lambda$ class). *For $v(z) = ze^{i\mu z}$ ($\mu > 0$), $\Phi(z) = E_1(i\mu z)$, and $c = -1$:*

$$\phi(z) = -E_1(i\mu z) + a + i\Omega_\gamma = \mathrm{Ci}(\mu z) + a + i\left(\mathrm{Si}(\mu z) - \frac{\pi}{2} + \Omega_\gamma\right).$$

For $a = 0$ and initial condition $\phi(u_0) = E_1(i\mu u_0)$, this recovers exactly the linearization of $\mathcal{F}_\lambda$ from [1], with $\Omega_\gamma = \pi/2 - \mathrm{Si}(\mu u_0)$.

Class 2 — Linear equation $\partial_v \phi = \alpha\,\mathrm{Re}(\phi) + \beta$

Proposition 12 (Linear v-ODE). *The v-ODE $\partial_v \phi = \alpha\,\mathrm{Re}(\phi) + \beta$, $\alpha \neq 0$, $\beta \in \mathbb{R}$, has unique solution*

$$\phi(z) = \left(P_0 + \frac{\beta}{\alpha}\right) e^{\alpha\Phi(z)} - \frac{\beta}{\alpha} + i\Omega_\gamma, \tag{10}$$

where $P_0 = \mathrm{Re}(\phi(z_0)) - \beta/\alpha$ and $\Omega_\gamma = \mathrm{Im}(\phi(z_0))$.

Proof. The reduced equation on $H(w) = \mathrm{Re}(G(w))$ is $H'(w) = \alpha H(w) + \beta$, a real linear first-order ODE with solution $H(w) = (H(w_0) + \beta/\alpha)e^{\alpha(w-w_0)} - \beta/\alpha$. Recomposing gives (10). □

Class 3 — Separable equation $\partial_v \phi = f(\mathrm{Re}(\phi)) \cdot g(z)$

Proposition 13 (Separable v-ODE). *The v-ODE $\partial_v \phi = f(\mathrm{Re}(\phi)) \cdot g(z)$, where $f : \mathbb{R} \to \mathbb{R}$ and $g : U \to \mathbb{R}$, reduces to the separated real equation*

$$\int_{P_0}^{P(z)} \frac{ds}{f(s)} = \int_{w_0}^{\Phi(z)} g(\Phi^{-1}(\xi))\,d\xi = \int_{z_0}^{z} g\,d_v z', \tag{11}$$

where $P(z) = \mathrm{Re}(\phi(z))$, $P_0 = \mathrm{Re}(\phi(z_0))$, and the right-hand side is the v-path integral of g.

Proof. Setting $H = \mathrm{Re}(G)$ in the reduced equation gives $H'(w) = f(H(w)) \cdot g(\Phi^{-1}(w))$, which separates as $dH/f(H) = g(\Phi^{-1}(w))\,dw$. Integrating and using Definition 4 for the right side gives (11). □

Remark 14. *The right-hand side of (11) is a v-path integral — the integration theory of Section 3 appears naturally in the solution of separable v-ODEs. The two parts of the calculus are therefore not independent.*

Class 4 — The A_n hierarchy as constant v_n-equations

Theorem 7 (A_n hierarchy as v_n-ODEs). *For each $n \geq 1$, let $v_n(z) = z^n e^{\lambda z}$ and $\Phi_n(z) = \lambda^{n-1} E_n(\lambda z)$. The v_n-ODE*

$$\partial_{v_n} \phi = -1 \tag{12}$$

has solution

$$\phi(z) = -\lambda^{n-1} E_n(\lambda z) + a + i\Omega_\gamma^{(n)}, \quad a \in \mathbb{R}, \tag{13}$$

where $\Omega_\gamma^{(n)} = \mathrm{Im}(\phi(z_0))$ is the topological invariant of the A_n hierarchy computed in [1]. The entire A_n hierarchy is therefore the family of solutions of the single equation type $\partial_{v_n} \phi = -1$, for varying $n \geq 1$.

Proof. Direct application of Proposition 11 with $c = -1$ and $\Phi = \Phi_n = \lambda^{n-1} E_n(\lambda z)$. □

Remark 15 (Unification of the hierarchy). *Theorem 7 is the deepest connection between the v-calculus and the $\mathcal{F}_\lambda$ programme. The entire hierarchy $A_1, A_2, A_3, \ldots$ — each with its own linearizer E_n, its own blowup time $\lambda^{n-1} E_n(\lambda u_0)$, and its own topological invariant $\Omega_\gamma^{(n)}$ — is the solution set of a single equation $\partial_{v_n} \phi = -1$, varying only the field v_n. The richness of the hierarchy is entirely encoded in the family of fields $v_n(z) = z^n e^{\lambda z}$ and their generating primitives $\lambda^{n-1} E_n$.*

7.3 The maximal bifurcation theorem

The following theorem collects the invisibility of Ω_γ across all operations of the v-calculus.

Theorem 8 (Maximal bifurcation). *Let $\phi \in \mathcal{D}_v(U)$ and write $\phi = \phi_{\mathrm{dyn}} + i\Omega_\gamma$ where $\phi_{\mathrm{dyn}} = \mathrm{Re}(\phi)$. Then:*

(i) ***Differentiation:*** *$\partial_v \phi = \partial_v \phi_{\mathrm{dyn}} \in \mathbb{R}$; the topological part Ω_γ contributes zero.*

(ii) ***Integration:*** *$\int_\gamma \partial_v \phi \, d_v t = \mathrm{Re}(\phi(\gamma(T))) - \mathrm{Re}(\phi(\gamma(0)))$; only the dynamic part is integrated.*

(iii) ***Taylor expansion:*** *all coefficients $\partial_v^n \phi(z_0)/n! \in \mathbb{R}$ belong to the dynamic part; Ω_γ appears only as the additive constant $i\Omega_\gamma$, irrecoverable from any derivative.*

(iv) ***Differential equations:*** *the v-ODE $\partial_v \phi = F(\phi, z)$ constrains only ϕ_{dyn}; Ω_γ is a free parameter determined solely by the initial condition.*

In particular, no operation of the v-calculus can create, modify, or detect Ω_γ. It is determined once and for all by the initial condition $\phi(z_0)$ and preserved exactly throughout all subsequent dynamics.

Proof. (i) follows from Definition 1; (ii) from Theorem 1; (iii) from Theorem 4; (iv) from Theorem 6. □

Operation	Dynamic part $\mathrm{Re}(\phi)$	Topological part Ω_γ
∂_v	transformed	invisible
$\int d_v t$	integrated	invisible
v-Taylor	expanded	invisible (additive constant only)
v-ODE	constrained	free parameter
Initial condition	fixes $\mathrm{Re}(\phi(z_0))$	fixes Ω_γ

8 Extension to $\mathbb{C}^n$

8.1 Definitions in dimension n

Let $U \subset \mathbb{C}^n$ be an open set and $v : U \to \mathbb{C}^n \setminus \{0\}$ a smooth nonvanishing field, $v = (v_1, \dots, v_n)$.

Definition 9 (v-differentiability in $\mathbb{C}^n$). *An analytic function $\phi : U \to \mathbb{C}$ is* v-differentiable *if*

$$d\phi(v)(z) := \sum_{k=1}^{n} \frac{\partial \phi}{\partial z_k}(z) \cdot v_k(z) \in \mathbb{R} \quad \text{for all } z \in U.$$

The v-derivative *is* $\partial_v \phi(z) := d\phi(v)(z) \in \mathbb{R}$.

Remark 16 (Scalar constraint on a vector). *In dimension* 1, $d\phi(v) = \phi' v$ *fixes the ratio* ϕ'/v *completely. In dimension* $n > 1$, *the condition* $d\phi(v) \in \mathbb{R}$ *is a single real constraint on the complex gradient* $\nabla\phi \in \mathbb{C}^n$, *leaving an* $(n-1)$*-complex-dimensional freedom in the directions orthogonal to* v. *The space* $\mathcal{D}_v(U)$ *is therefore much richer in higher dimensions.*

Proposition 14 (Algebraic structure in $\mathbb{C}^n$). *All algebraic results of Section 2 hold verbatim in dimension* n: $\mathcal{D}_v(U)$ *is a real vector space, not a* $\mathbb{C}$*-module, with the same Leibniz constraint and chain rule.*

Proof. The proofs of Propositions 1–4 use only linearity of $d\phi(v)$ in ϕ and the product rule, both valid in any dimension. □

8.2 The generating form and integrability

Definition 10 (Generating form). *To the field* v *we associate the holomorphic* 1*-form*

$$\omega_v := \sum_{k=1}^{n} v_k(z)\, dz_k.$$

The v*-differentiability condition reads* $\omega_v(\nabla\phi) \in \mathbb{R}$.

Definition 11 (Integrable field in $\mathbb{C}^n$). *The field* v *is* integrable *if* ω_v *is closed:*

$$d\omega_v = 0 \iff \frac{\partial v_j}{\partial z_k} = \frac{\partial v_k}{\partial z_j} \quad \text{for all } j, k.$$

By the Poincaré lemma, on a simply connected U *this is equivalent to the existence of an analytic* generating primitive $\Phi : U \to \mathbb{C}$ *with* $d\Phi = \omega_v$, *i.e.,* $\partial\Phi/\partial z_k = v_k$ *for all* k.

Definition 12 (Normalized field). *An integrable field* v *with primitive* Φ *is* normalized *if*

$$\omega_v(v)(z) = \sum_{k=1}^{n} v_k(z)^2 = 1 \quad \text{for all } z \in U.$$

In dimension 1, *normalization was automatic by construction of* $d\Phi = dz/v$. *In dimension* n, *it is an independent condition.*

Proposition 15 (Richness under integrability in $\mathbb{C}^n$). *If* v *is integrable and normalized with primitive* Φ, *then for any analytic* F *with* F' *real-valued,* $F \circ \Phi \in \mathcal{D}_v(U)$ *and* $\partial_v(F \circ \Phi)(z) = F'(\Phi(z)) \in \mathbb{R}$.

Proof. $d(F \circ \Phi)(v) = F'(\Phi(z)) \cdot d\Phi(v) = F'(\Phi(z)) \cdot \omega_v(v) = F'(\Phi(z)) \cdot 1 \in \mathbb{R}$. □

8.3 The dimensional reduction theorem

Theorem 9 (Dimensional reduction). *Let v be integrable and normalized on $U \subset \mathbb{C}^n$ with generating primitive $\Phi : U \to \mathbb{C}$. For $\phi = F \circ \Phi \in \mathcal{D}_v(U)$:*

$$\partial_v \phi(z) = F'(\Phi(z)). \tag{14}$$

The v-calculus on $U \subset \mathbb{C}^n$ is isomorphic to the standard real calculus on $\Phi(U) \subset \mathbb{C}$ via $w = \Phi(z)$. In particular, the iterated derivatives, v-Taylor series, v-ODEs, and maximal bifurcation theorem all hold in dimension n with the same statements and proofs as in dimension 1.

Proof. $\partial_v(F \circ \Phi)(z) = d(F \circ \Phi)(v)(z) = F'(\Phi(z)) \cdot d\Phi(v)(z) = F'(\Phi(z)) \cdot \omega_v(v)(z) = F'(\Phi(z))$ by normalization. □

Corollary 7 (Conservation of Ω_γ in $\mathbb{C}^n$). *Let γ be an integral curve of v in $U \subset \mathbb{C}^n$ and $\phi \in \mathcal{D}_v(U)$. Then $\Omega_\gamma = \mathrm{Im}(\phi(\gamma(t))) - 2\pi\mathcal{K}(t)$ is conserved along γ.*

Proof. $\frac{d}{dt}\phi(\gamma(t)) = d\phi(v(\gamma(t))) = \partial_v\phi(\gamma(t)) \in \mathbb{R}$, so $\mathrm{Im}(\phi(\gamma(t)))$ is constant between branch cuts. □

8.4 The genuine novelty: fibre topology

Theorem 9 shows the v-calculus always reduces to dimension 1. The genuine n-dimensional novelty lies in the fibres.

Definition 13 (Fibres of Φ). *For $c \in \Phi(U)$, the* fibre *is $\mathcal{F}_c := \{z \in U : \Phi(z) = c\}$, a complex hypersurface of dimension $n-1$ in U.*

Proposition 16 (Transversal crossing of fibres). *Along any integral curve γ of v: $\frac{d}{dt}\Phi(\gamma(t)) = \omega_v(v)(\gamma(t)) = 1$. Hence $\Phi(\gamma(t)) = \Phi(\gamma(0)) + t$: integral curves cross fibres at unit speed, transversally.*

Proof. $\frac{d}{dt}\Phi(\gamma(t)) = d\Phi(\dot\gamma(t)) = d\Phi(v(\gamma(t))) = \omega_v(v)(\gamma(t)) = 1$ by normalization. □

Remark 17 (What lives on fibres). *On a single fibre $\mathcal{F}_c$, the v-derivative $\partial_v\phi$ vanishes identically (since Φ is constant there). The restriction of ϕ to $\mathcal{F}_c$ is the initial data of the v-calculus, and Ω_γ depends only on the value of ϕ on $\mathcal{F}_{\Phi(z_0)}$. The topology of $\mathcal{F}_c$ — its homology, fundamental group, characteristic classes — is the repository of all genuinely n-dimensional information invisible to the one-dimensional reduction.*

8.5 Dimension 1 vs dimension n: comparison

Feature	Dimension 1	Dimension n
Field v	$v : U \to \mathbb{C} \setminus \{0\}$	$v : U \to \mathbb{C}^n \setminus \{0\}$
Integrability	$1/v$ has analytic primitive	$\omega_v = \sum v_k\, dz_k$ closed
Normalization	automatic	requires $\omega_v(v) = 1$
Primitive Φ	$\Phi : U \to \mathbb{C}$, scalar	$\Phi : U \to \mathbb{C}$, still scalar
Reduction	to $\mathbb{R}$-calculus via Φ	same
Ω_γ	conserved	conserved (same proof)
Maximal bifurcation	holds	holds (same proof)
Fibres of Φ	discrete points	hypersurfaces in $\mathbb{C}^{n-1}$
New topology	residues of $1/v$	fibre topology + residues of ω_v

9 v-Elementary Functions

9.1 Construction principle

In the v-calculus, every v-differentiable function has the form $\phi = F \circ \Phi$ where Φ is the generating primitive and F' is real-valued. This gives a universal construction principle for a complete family of elementary functions.

Definition 14 (v-elementary function). *A function $\phi : U \to \mathbb{C}$ is* v-elementary *if $\phi = F \circ \Phi$ where F is a classical elementary function with F' real-valued on $\Phi(U)$.*

Every classical elementary function F with real derivative produces a v-elementary function via $\phi = F \circ \Phi$. The v-calculus inherits a complete family of elementary functions from the classical calculus, transported through the geometry of Φ.

Throughout this section we work with the canonical field $v(z) = ze^{\lambda z}$, $\Phi(z) = E_1(\lambda z)$, the unique field of the $\mathcal{F}_\lambda$ class [1] with residue 1 at $z = 0$ and transcendental non-elementary linearizer.

9.2 The v-exponential

Definition 15 (v-exponential). *The* v-exponential *is*

$$\mathbf{E}_v(z) := e^{E_1(\lambda z)}, \quad z \in \mathbb{C} \setminus \{0\}. \tag{15}$$

It corresponds to $F(w) = e^w$ in the construction $\phi = F \circ \Phi$.

Theorem 10 (Eigenequation). *The v-exponential satisfies $\partial_v \mathbf{E}_v = \mathbf{E}_v$. It is the unique v-differentiable function (up to real scalar and additive imaginary constant) satisfying this equation.*

Proof. By the chain rule of the v-calculus (Proposition 4): $\partial_v \mathbf{E}_v(z) = (e^w)'|_{w=E_1(\lambda z)} \cdot \partial_v(E_1(\lambda z)) = e^{E_1(\lambda z)} \cdot 1 = \mathbf{E}_v(z)$. Uniqueness: the reduced equation $G'(w) = G(w)$ has unique solution $G(w) = Ce^w$, $C \in \mathbb{R}$, giving $\phi = C\mathbf{E}_v + i\Omega_\gamma$. □

Proposition 17 (Dynamical-topological decomposition). *For $\lambda = i\mu$ ($\mu > 0$) and $z = u \in \mathbb{R}_{>0}$:*

$$\mathbf{E}_v(u) = \underbrace{e^{-\mathrm{Ci}(\mu u)}}_{\text{amplitude (dynamic)}} \cdot \underbrace{e^{i(\pi/2 - \mathrm{Si}(\mu u))}}_{\text{phase (topological)}}. \tag{16}$$

The amplitude $e^{-\mathrm{Ci}(\mu u)} > 0$ is the dynamic part, evolved by ∂_v. The phase $e^{i\Omega_\gamma}$ with $\Omega_\gamma = \pi/2 - \mathrm{Si}(\mu u)$ is the topological invariant, frozen and invisible to ∂_v.

Proof. Use $E_1(i\mu u) = -\mathrm{Ci}(\mu u) + i(\pi/2 - \mathrm{Si}(\mu u))$ and exponentiate. □

Proposition 18 (Limit behavior — duality with classical exponential). *(i) As $z \to 0^+$:* $\mathrm{Re}(E_1(\lambda z)) \to +\infty$, *so $|\mathbf{E}_v(z)| \to +\infty$.*

(ii) As $z \to +\infty$: $E_1(\lambda z) \to 0$, so $\mathbf{E}_v(z) \to 1$.

The v-exponential decays from $+\infty$ to 1, in contrast to the classical exponential e^x which grows from 0 to $+\infty$. This duality reflects the structure of E_1: large near 0 (where v vanishes) and small at $+\infty$.

Proposition 19 (v-exponential as exact propagator). *The solution of the v-ODE $\partial_v \phi = \phi$ with initial condition $\phi(z_0) = \phi_0 \in \mathbb{C}$ is:*

$$\phi(z) = \mathrm{Re}(\phi_0) \cdot \frac{\mathbf{E}_v(z)}{\mathbf{E}_v(z_0)} + i\,\mathrm{Im}(\phi_0). \tag{17}$$

The real part propagates via $\mathbf{E}_v$; the imaginary part $\Omega_\gamma = \mathrm{Im}(\phi_0)$ is frozen.

Proof. The reduced equation $G'(w) = G(w)$ with $G(w_0) = \mathrm{Re}(\phi_0)$ gives $G(w) = \mathrm{Re}(\phi_0) \cdot e^{w-w_0}$. Recomposing: $\phi(z) = \mathrm{Re}(\phi_0) \cdot e^{E_1(\lambda z) - E_1(\lambda z_0)} + i\Omega_\gamma = \mathrm{Re}(\phi_0) \cdot \mathbf{E}_v(z)/\mathbf{E}_v(z_0) + i\Omega_\gamma$. □

9.3 v-Trigonometric functions

Definition 16 (v-sine and v-cosine).

$$\mathbf{S}_v(z) := \sin(E_1(\lambda z)), \qquad \mathbf{C}_v(z) := \cos(E_1(\lambda z)).$$

Proposition 20 (v-trigonometric calculus).

$$\partial_v \mathbf{S}_v = \mathbf{C}_v, \qquad \partial_v \mathbf{C}_v = -\mathbf{S}_v, \qquad \mathbf{S}_v(z)^2 + \mathbf{C}_v(z)^2 = 1.$$

Proof. $\partial_v \mathbf{S}_v(z) = \cos(E_1(\lambda z)) \cdot \partial_v(E_1(\lambda z)) = \mathbf{C}_v(z) \cdot 1$. Similarly for $\mathbf{C}_v$. The Pythagorean identity follows from the classical $\sin^2 + \cos^2 = 1$ evaluated at $E_1(\lambda z)$. □

Remark 18 (v-harmonic oscillator). *Since $\partial_v^2 \mathbf{S}_v = \partial_v \mathbf{C}_v = -\mathbf{S}_v$ and $\partial_v^2 \mathbf{C}_v = -\partial_v \mathbf{S}_v = -\mathbf{C}_v$, both $\mathbf{S}_v$ and $\mathbf{C}_v$ satisfy the v-ODE*

$$\partial_v^2 \phi = -\phi,$$

the v-harmonic oscillator equation. *This is the exact v-analogue of $f'' = -f$ whose classical solutions are* sin *and* cos.

9.4 v-Power functions and v-logarithm

Definition 17 (v-powers and v-logarithm). *For $n \geq 0$:*

$$\mathbf{P}_v^n(z) := E_1(\lambda z)^n, \qquad \mathbf{L}_v(z) := \ln(E_1(\lambda z)).$$

Proposition 21 (v-power and v-logarithm derivatives).

$$\partial_v \mathbf{P}_v^n = n\,\mathbf{P}_v^{n-1}, \qquad \partial_v \mathbf{L}_v = \frac{1}{\mathbf{P}_v^1}.$$

The v-powers $\{\mathbf{P}_v^n\}_{n\geq 0}$ are the natural basis for v-Taylor expansions (Theorem 3).

Proof. $\partial_v(E_1(\lambda z)^n) = n\,E_1(\lambda z)^{n-1} \cdot 1 = n\mathbf{P}_v^{n-1}$. $\partial_v(\ln E_1(\lambda z)) = (1/E_1(\lambda z)) \cdot 1 = 1/\mathbf{P}_v^1$. □

9.5 The v_n-exponential hierarchy

Proposition 22 (v_n-exponentials from the A_n hierarchy). *For each $n \geq 1$, with $v_n(z) = z^n e^{\lambda z}$ and $\Phi_n(z) = \lambda^{n-1} E_n(\lambda z)$, define*

$$\mathbf{E}_{v_n}(z) := e^{\lambda^{n-1} E_n(\lambda z)}.$$

Then $\partial_{v_n} \mathbf{E}_{v_n} = \mathbf{E}_{v_n}$, and the topological invariant is $\Omega_\gamma^{(n)}$ from [1]. The entire A_n hierarchy produces a coherent tower of v-exponentials, generated by $\hat{L} = \partial_\lambda$ acting on $v_1 = ze^{\lambda z}$.

Proof. $\partial_{v_n} \mathbf{E}_{v_n}(z) = e^{\Phi_n(z)} \cdot \partial_{v_n}(\Phi_n(z)) = \mathbf{E}_{v_n}(z) \cdot 1 = \mathbf{E}_{v_n}(z)$, since $\partial_{v_n} \Phi_n = 1$ by normalization. □

9.6 Summary table

Function	Definition	$\partial_v \phi$	v-ODE	Classical analogue
v-exponential	$e^{E_1(\lambda z)}$	$\mathbf{E}_v$	$\partial_v \phi = \phi$	e^x
v-sine	$\sin(E_1(\lambda z))$	$\mathbf{C}_v$	$\partial_v^2 \phi = -\phi$	$\sin x$
v-cosine	$\cos(E_1(\lambda z))$	$-\mathbf{S}_v$	$\partial_v^2 \phi = -\phi$	$\cos x$
v-power n	$E_1(\lambda z)^n$	$n\mathbf{P}_v^{n-1}$	v-Taylor basis	x^n
v-logarithm	$\ln(E_1(\lambda z))$	$1/\mathbf{P}_v^1$	—	$\ln x$

9.7 The v-Euler identity

In classical analysis, the Euler identity $e^{ix} = \cos x + i \sin x$ connects the real exponential, the cosine, and the sine through the imaginary unit. We now derive the exact analogue in the v-calculus and identify when the two v-exponential objects coincide.

Set $W := E_1(\lambda z) \in \mathbb{C}$. The two natural complex exponentials built from W are:

$$\mathbf{E}_v(z) = e^W \qquad \text{and} \qquad \mathbf{C}_v(z) + i\mathbf{S}_v(z) = \cos W + i \sin W = e^{iW}.$$

These are genuinely distinct objects: $\mathbf{E}_v$ exponentiates W directly, while $\mathbf{C}_v + i\mathbf{S}_v$ exponentiates iW.

Theorem 11 (v-Euler identity). *For all $z \in \mathbb{C} \setminus \{0\}$ and $\lambda > 0$:*

$$\boxed{\mathbf{E}_v(z) = e^{(1-i)\, E_1(\lambda z)} \cdot \big(\mathbf{C}_v(z) + i\mathbf{S}_v(z)\big).} \tag{18}$$

The factor $e^{(1-i)E_1(\lambda z)}$ is the unique corrector relating the two v-exponential objects. Equivalently:

$$\ln \mathbf{E}_v(z) - \ln(\mathbf{C}_v(z) + i\mathbf{S}_v(z)) = (1 - i)\, E_1(\lambda z). \tag{19}$$

Proof. Direct computation: $e^W / e^{iW} = e^{W - iW} = e^{(1-i)W}$, so $e^W = e^{(1-i)W} \cdot e^{iW}$. Substituting $W = E_1(\lambda z)$ and $e^{iW} = \mathbf{C}_v(z) + i\mathbf{S}_v(z)$ gives (18). □

Corollary 8 (Classical Euler as special case). *The classical Euler identity $e^{ix} = \cos x + i \sin x$ is recovered when $W = ix$ for $x \in \mathbb{R}$, giving $e^{(1-i)\cdot ix} = e^{ix+x} = e^x \cdot e^{ix}$... This shows that the classical identity does not embed directly into the v-calculus without rescaling. The v-Euler identity (18) is the correct generalization, reducing to the classical one only when $E_1(\lambda z) \in \mathbb{R}$ (i.e., $\lambda > 0$, $z > 0$), in which case $e^{(1-i)E_1} = e^{(1-i)r}$ for $r \in \mathbb{R}_{>0}$, and the corrector is non-trivial unless $r = 0$.*

Proposition 23 (Coincidence condition). $\mathbf{E}_v(z) = \mathbf{C}_v(z) + i\mathbf{S}_v(z)$ *if and only if* $E_1(\lambda z) \in \pi(1+i)\mathbb{Z}$, *i.e.,* $E_1(\lambda z) = \pi(1+i)k$ *for some* $k \in \mathbb{Z}$.

Proof. $e^W = e^{iW} \iff e^{W(1-i)} = 1 \iff W(1-i) \in 2\pi i\mathbb{Z} \iff W \in \frac{2\pi i}{1-i}\mathbb{Z} = \pi(1+i)\mathbb{Z}$. □

Remark 19 (Decomposition of the corrector). *Write* $W = -\mathrm{Ci}(\mu z) + i\Omega_\gamma$ *for* $\lambda = i\mu$. *Then:*

$$e^{(1-i)W} = e^{(1-i)(-\mathrm{Ci}(\mu z)+i\Omega_\gamma)} = e^{-\mathrm{Ci}(\mu z)+\Omega_\gamma} \cdot e^{-i(\mathrm{Ci}(\mu z)+\Omega_\gamma)}.$$

The corrector has amplitude $e^{-\mathrm{Ci}(\mu z)+\Omega_\gamma}$ *and phase* $-(\mathrm{Ci}(\mu z)+\Omega_\gamma)$*: it mixes the dynamic part* $\mathrm{Ci}(\mu z)$ *and the topological part* Ω_γ *in both its amplitude and its phase simultaneously. This mixing is the signature of the v-Euler identity — absent in the classical case where* Ω_γ *does not exist.*

Proposition 24 (Amplitude-phase exchange). *In the decomposition* $\mathbf{E}_v(z) = |\mathbf{E}_v| \cdot e^{i\arg(\mathbf{E}_v)}$ *and* $\mathbf{C}_v(z) + i\mathbf{S}_v(z) = |\mathbf{C}_v + i\mathbf{S}_v| \cdot e^{i\arg(\mathbf{C}_v+i\mathbf{S}_v)}$*:*

$$|\mathbf{E}_v(z)| = e^{-\mathrm{Ci}(\mu z)}, \qquad \arg(\mathbf{E}_v(z)) = \Omega_\gamma, \tag{20}$$

$$|\mathbf{C}_v(z) + i\mathbf{S}_v(z)| = e^{\Omega_\gamma}, \qquad \arg(\mathbf{C}_v(z) + i\mathbf{S}_v(z)) = -\mathrm{Ci}(\mu z). \tag{21}$$

The dynamic part $\mathrm{Ci}(\mu z)$ *and the topological part* Ω_γ *play* exchanged roles *in* $\mathbf{E}_v$ *and* $\mathbf{C}_v + i\mathbf{S}_v$*: what is amplitude in one is phase in the other.*

Proof. For $\lambda = i\mu$: $E_1(i\mu z) = -\mathrm{Ci}(\mu z) + i\Omega_\gamma$. Then $\mathbf{E}_v = e^{-\mathrm{Ci}+i\Omega_\gamma}$ gives amplitude $e^{-\mathrm{Ci}}$ and phase Ω_γ. And $\mathbf{C}_v + i\mathbf{S}_v = e^{i(-\mathrm{Ci}+i\Omega_\gamma)} = e^{-\Omega_\gamma - i\,\mathrm{Ci}}$ gives amplitude $e^{-\Omega_\gamma}$ and phase $-\mathrm{Ci}$. □

Remark 20 (The v-Euler identity as amplitude-phase duality). *Proposition 24 reveals the geometric meaning of the v-Euler identity: the two v-exponential objects* $\mathbf{E}_v$ *and* $\mathbf{C}_v + i\mathbf{S}_v$ *are* dual *in the sense that their amplitude and phase roles are swapped. This duality is encoded in the corrector* $e^{(1-i)E_1(\lambda z)}$*, which performs exactly the exchange. In classical analysis this duality is invisible because* Ω_γ *does not exist:* e^x *and* e^{ix} *differ only by one being real and the other being on the unit circle. The v-calculus reveals a richer structure behind this classical dichotomy.*

Open Question 1 (v-Fourier analysis). *The v-trigonometric functions* $\mathbf{S}_v$ *and* $\mathbf{C}_v$ *satisfy* $\partial_v\mathbf{S}_v = \mathbf{C}_v$, $\partial_v\mathbf{C}_v = -\mathbf{S}_v$, *and the Pythagorean identity. Is there a v-Fourier theory — expansions of v-differentiable functions in series of* $\mathbf{S}_v$ *and* $\mathbf{C}_v$ *— analogous to classical Fourier analysis? What is the v-analogue of the Fourier transform, and what is the orthogonality relation replacing* $\int_0^{2\pi} \sin mx \sin nx\, dx$*?*

Open Question 2 (Tower of v_n-elementary functions). *For each* $n \geq 1$*, the field* v_n *produces a family* $(\mathbf{E}_{v_n}, \mathbf{S}_{v_n}, \mathbf{C}_{v_n}, \ldots)$ *of* v_n*-elementary functions. Do these families form a coherent tower under* $\hat{L} = \partial_\lambda$*, and what is the algebraic structure of the tower? In particular, does* $\hat{L}(\mathbf{E}_{v_n}) = \mathbf{E}_{v_{n+1}}$ *hold in some sense?*

Open Question 3 (Fibre topology and Ω_γ). *The topological invariant* Ω_γ *is determined by* ϕ *on the initial fibre* $\mathcal{F}_{\Phi(z_0)}$*. In dimension* $n \geq 2$*, fibres are complex hypersurfaces with non-trivial topology. Is there a precise correspondence between the topology of* $\mathcal{F}_c$ *(homology,* π_1*, characteristic classes) and the admissible values of* Ω_γ*? In particular: does a non-trivial* $\pi_1(\mathcal{F}_c)$ *force* $\Omega_\gamma \in 2\pi\mathbb{Z}$*?*

Open Question 4 (Cohomological interpretation of Ω_γ). *The invariant Ω_γ is locally constant along γ and jumps by $2\pi\mathbb{Z}$ at branch cuts, suggesting it defines a class in $H^1(U,\mathbb{Z})$. Is there a natural sheaf or de Rham cohomology theory on U in which $[\Omega_\gamma]$ is a well-defined class, and in which the residue structure of ω_v determines the cohomology?*

Open Question 5 (v-calculus on complex manifolds). *Can the v-calculus be extended to complex manifolds M, where v is a holomorphic vector field and Φ a holomorphic function? The global obstruction to the existence of Φ would live in $H^1(M,\mathcal{O})$, connecting the v-calculus to classical complex geometry and the Dolbeault cohomology.*

10 Concrete Calculations in the v-Calculus

We work throughout with the canonical field $v(z) = ze^{\lambda z}$, $\Phi(z) = E_1(\lambda z)$, $\lambda > 0$, $z > 0$ real. All computations use the reduction $\partial_v^n \phi(z) = F^{(n)}(E_1(\lambda z))$ where $\phi = F \circ \Phi$.

10.1 v-Derivatives

Example 2 (v-power). *Let $\phi(z) = E_1(\lambda z)^3 = \mathbf{P}_v^3(z)$, i.e., $F(w) = w^3$.*

$$\partial_v \phi(z) = F'(E_1(\lambda z)) = 3\,E_1(\lambda z)^2 = 3\mathbf{P}_v^2(z).$$

Example 3 (Scaled v-exponential). *Let $\phi(z) = e^{2E_1(\lambda z)}$, i.e., $F(w) = e^{2w}$.*

$$\partial_v \phi(z) = 2e^{2E_1(\lambda z)} = 2\phi(z).$$

This function satisfies the eigenequation $\partial_v \phi = 2\phi$: it is the v-exponential with eigenvalue 2.

Example 4 (Composed v-function). *Let $\phi(z) = \sin(E_1(\lambda z)^2)$, i.e., $F(w) = \sin(w^2)$.*

$$F'(w) = 2w\cos(w^2), \qquad \partial_v \phi(z) = 2\,E_1(\lambda z)\cos(E_1(\lambda z)^2) = 2\mathbf{P}_v^1(z)\cdot\cos(\mathbf{P}_v^2(z)).$$

10.2 v-Primitives

Example 5 (v-primitive of $\mathbf{P}_v^2$). *Find Φ_f such that $\partial_v \Phi_f = \mathbf{P}_v^2(z) = E_1(\lambda z)^2$. We need $F'(w) = w^2$, so $F(w) = w^3/3$:*

$$\Phi_f(z) = \frac{E_1(\lambda z)^3}{3} = \frac{\mathbf{P}_v^3(z)}{3}.$$

Verification: $\partial_v(\mathbf{P}_v^3/3) = 3\mathbf{P}_v^2/3 = \mathbf{P}_v^2$. ✓

Example 6 (v-primitive of $\mathbf{E}_v$). *Find Φ_f such that $\partial_v \Phi_f = \mathbf{E}_v(z) = e^{E_1(\lambda z)}$. We need $F'(w) = e^w$, so $F(w) = e^w$:*

$$\Phi_f(z) = e^{E_1(\lambda z)} = \mathbf{E}_v(z).$$

The v-exponential is its own v-primitive — exactly as $\int e^x\,dx = e^x$.

Example 7 (v-primitive of $\mathbf{C}_v$). *Find Φ_f such that $\partial_v \Phi_f = \mathbf{C}_v(z) = \cos(E_1(\lambda z))$. We need $F'(w) = \cos w$, so $F(w) = \sin w$:*

$$\Phi_f(z) = \sin(E_1(\lambda z)) = \mathbf{S}_v(z).$$

The v-primitive of $\mathbf{C}_v$ is $\mathbf{S}_v$ — exactly as $\int \cos x\,dx = \sin x$.

10.3 A complete study: $\phi(z) = \sin(E_1(\lambda z)^2)$

We carry out a full analysis of the function $\phi(z) = \sin(E_1(\lambda z)^2)$, i.e., $F(w) = \sin(w^2)$, with $v(z) = ze^{\lambda z}$, $\Phi(z) = E_1(\lambda z)$, $\lambda > 0$, $z > 0$ real. In v-power notation: $\phi(z) = \sin(\mathbf{P}_v^2(z))$. This example is the richest of the concrete examples: it combines v-power and v-trigonometric structure, and all phenomena of the calculus appear.

v-Derivative

By the chain rule (Proposition 4) and $F'(w) = 2w\cos(w^2)$:

$$\partial_v \phi(z) = F'(E_1(\lambda z)) = 2\,E_1(\lambda z) \cdot \cos(E_1(\lambda z)^2) = 2\,\mathbf{P}_v^1(z) \cdot \cos(\mathbf{P}_v^2(z)). \tag{22}$$

Iterated v-derivatives

By Theorem 2, $\partial_v^n \phi(z) = F^{(n)}(E_1(\lambda z))$. Computing $F''(w) = 2\cos(w^2) - 4w^2\sin(w^2)$ and $F'''(w) = -12w\cos(w^2) + 8w^3\sin(w^2)$:

$$\partial_v^2 \phi(z) = 2\cos(E_1(\lambda z)^2) - 4\,E_1(\lambda z)^2 \cdot \sin(E_1(\lambda z)^2) = 2\cos(\mathbf{P}_v^2) - 4\mathbf{P}_v^2 \cdot \phi, \tag{23}$$

$$\partial_v^3 \phi(z) = -12\,\mathbf{P}_v^1 \cdot \cos(\mathbf{P}_v^2) + 8\,(\mathbf{P}_v^1)^3 \cdot \phi = -6\partial_v \phi + 8(\mathbf{P}_v^1)^2 \cdot \phi. \tag{24}$$

All iterated derivatives express in terms of ϕ and $\cos(\mathbf{P}_v^2)$, exactly as in the real calculus: $(\sin x^2)'' = 2\cos x^2 - 4x^2 \sin x^2$.

v-Taylor series

Setting $w_0 = E_1(\lambda z_0)$ and $\Delta w = E_1(\lambda z) - w_0$, the v-Taylor series at z_0 is:

$$\begin{aligned} \phi(z) = \sin(w_0^2) &+ 2w_0\cos(w_0^2) \cdot \Delta w \\ &+ \left(\cos(w_0^2) - 2w_0^2\sin(w_0^2)\right) \cdot \Delta w^2 \\ &+ \frac{-12w_0\cos(w_0^2) + 8w_0^3\sin(w_0^2)}{6} \cdot \Delta w^3 + \cdots + i\Omega_\gamma, \end{aligned} \tag{25}$$

with all coefficients in $\mathbb{R}$ and radius of convergence $R_v = +\infty$ (since $F(w) = \sin(w^2)$ is entire).

v-Primitive and the Fresnel integral

We seek Φ_f such that $\partial_v \Phi_f = \phi(z) = \sin(E_1(\lambda z)^2)$. This requires $G'(w) = \sin(w^2)$, whose primitive is the *Fresnel sine integral*:

$$G(w) = \int_0^w \sin(t^2)\,dt = \sqrt{\frac{\pi}{2}}\,S\left(\sqrt{\frac{2}{\pi}}\,w\right),$$

where $S(x) = \int_0^x \sin(\pi t^2/2)\,dt$ is the standard Fresnel function. Therefore:

$$\boxed{\Phi_f(z) = \sqrt{\frac{\pi}{2}}\,S\left(\sqrt{\frac{2}{\pi}}\,E_1(\lambda z)\right).} \tag{26}$$

Remark 21 (Unexpected connection to optics). *The v-primitive of $\sin(E_1(\lambda z)^2)$ is the Fresnel integral — a function governing light diffraction and wave propagation in optics. This connection was not anticipated: it emerges naturally from the structure of the v-calculus, linking the exponential integral E_1 to the Fresnel function S through the composition $F(w) = \sin(w^2)$.*

Definite v-integral and a universal constant

By the fundamental theorem, along an integral curve of v from z_0 to z_1:

$$\int_\gamma \phi \, d_v t = \sqrt{\frac{\pi}{2}} \left[S\left(\sqrt{\frac{2}{\pi}} \, E_1(\lambda z_1) \right) - S\left(\sqrt{\frac{2}{\pi}} \, E_1(\lambda z_0) \right) \right].$$

In the limit $z_0 \to +\infty$ (so $E_1(\lambda z_0) \to 0$, $S \to 0$) and $z_1 \to 0^+$ (so $E_1(\lambda z_1) \to +\infty$, $S \to 1/2$):

$$\int_\gamma \phi \, d_v t = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} = \frac{\sqrt{2\pi}}{4}. \tag{27}$$

Remark 22 (Universal constant). *The total v-integral of $\sin(E_1(\lambda z)^2)$ along a complete integral curve of v equals $\sqrt{2\pi}/4$, independently of λ. This is a universal constant of the v-calculus for this function — a structural invariant invisible to the parameter λ.*

v-ODE: $\partial_v \psi = \phi$

Solving $\partial_v \psi = \sin(E_1(\lambda z)^2)$ with $\psi(z_0) = \psi_0 \in \mathbb{R}$: the reduction gives $G'(w) = \sin(w^2)$, $G(w_0) = \psi_0$, hence:

$$\psi(z) = \psi_0 + \sqrt{\frac{\pi}{2}} \left[S\left(\sqrt{\frac{2}{\pi}} \, E_1(\lambda z) \right) - S\left(\sqrt{\frac{2}{\pi}} \, E_1(\lambda z_0) \right) \right] + i\Omega_\gamma. \tag{28}$$

Summary table

Operation	Result	Remark
$\partial_v \phi$	$2E_1 \cos(E_1^2)$	double chain rule
$\partial_v^2 \phi$	$2\cos(E_1^2) - 4E_1^2 \sin(E_1^2)$	expressible via ϕ
v-primitive	Fresnel integral S	connection to optics
Total v-integral	$\sqrt{2\pi}/4$	universal, independent of λ
v-ODE $\partial_v \psi = \phi$	solution via S	Fresnel function
Ω_γ ($\lambda = i\mu$)	$\pi/2 - \mathrm{Si}(\mu z_0)$	frozen, invisible to all

10.4 A second complete study: $\phi(z) = e^{E_1(\lambda z)^2}$

We carry out a full analysis of $\phi(z) = e^{E_1(\lambda z)^2}$, i.e., $F(w) = e^{w^2}$, with the same canonical field $v(z) = ze^{\lambda z}$, $\lambda > 0$, $z > 0$ real.

v-Derivative

$F'(w) = 2we^{w^2}$, so:

$$\partial_v \phi(z) = 2\, E_1(\lambda z) \cdot e^{E_1(\lambda z)^2} = 2\, \mathbf{P}_v^1(z) \cdot \phi(z). \tag{29}$$

This is a *generalized eigenequation*: $\partial_v \phi = 2\mathbf{P}_v^1 \cdot \phi$, in which the eigenvalue is not a constant but the function $2\mathbf{P}_v^1(z)$.

Iterated v-derivatives

$F''(w) = (4w^2 + 2)e^{w^2}$ and $F'''(w) = (8w^3 + 12w)e^{w^2}$:

$$\partial_v^2 \phi(z) = \left(4\,E_1(\lambda z)^2 + 2\right) e^{E_1(\lambda z)^2} = (4\mathbf{P}_v^2 + 2) \cdot \phi, \tag{30}$$

$$\partial_v^3 \phi(z) = \left(8\,E_1(\lambda z)^3 + 12\,E_1(\lambda z)\right) e^{E_1(\lambda z)^2} = (8(\mathbf{P}_v^1)^3 + 12\mathbf{P}_v^1) \cdot \phi. \tag{31}$$

All iterated derivatives factor through ϕ itself — a structural property of $F(w) = e^{w^2}$ propagated by Theorem 2.

v-Taylor series

Setting $w_0 = E_1(\lambda z_0)$, $\Delta w = E_1(\lambda z) - w_0$:

$$\phi(z) = e^{w_0^2} + 2w_0 e^{w_0^2} \cdot \Delta w + (2w_0^2 + 1)e^{w_0^2} \cdot \Delta w^2 + \cdots + i\Omega_\gamma, \tag{32}$$

with all coefficients in $\mathbb{R}$ and radius $R_v = +\infty$ (since e^{w^2} is entire).

v-Primitive and the error function

We seek Φ_f such that $\partial_v \Phi_f = e^{E_1(\lambda z)^2}$. This requires $G'(w) = e^{w^2}$, whose primitive is the *imaginary error function*:

$$G(w) = \int_0^w e^{t^2}\,dt = \frac{\sqrt{\pi}}{2}\,\mathrm{erfi}(w),$$

where $\mathrm{erfi}(w) = \frac{2}{\sqrt{\pi}} \int_0^w e^{t^2}\,dt$. Therefore:

$$\boxed{\Phi_f(z) = \frac{\sqrt{\pi}}{2}\,\mathrm{erfi}(E_1(\lambda z)).} \tag{33}$$

Remark 23 (Connection to the error function and probability). *The v-primitive of $e^{E_1(\lambda z)^2}$ is the imaginary error function* erfi. *For $\lambda = i\mu$, $E_1(i\mu z)$ is complex, and the identity* $\mathrm{erfi}(ix) = i\,\mathrm{erf}(x)$ *connects to the classical error function* erf *— the fundamental function of probability theory and the normal distribution. As with the Fresnel integral, this connection is not imposed: it emerges from the structure of $F(w) = e^{w^2}$ transported through $\Phi = E_1(\lambda z)$.*

Definite v-integral

Along an integral curve from z_0 to z_1:

$$\int_\gamma \phi\,d_v t = \frac{\sqrt{\pi}}{2}\left[\mathrm{erfi}(E_1(\lambda z_1)) - \mathrm{erfi}(E_1(\lambda z_0))\right].$$

As $z_1 \to 0^+$, $E_1(\lambda z_1) \to +\infty$ and $\mathrm{erfi}(+\infty) = +\infty$: the total integral diverges. This is not a defect — it reflects that $e^{E_1(\lambda z)^2}$ grows too rapidly near $z = 0$ to be integrable on the full curve. The divergence is a structural fact encoded in $F(w) = e^{w^2}$.

v-ODE: $\partial_v \psi = \phi$

Solving $\partial_v \psi = e^{E_1(\lambda z)^2}$ with $\psi(z_0) = \psi_0 \in \mathbb{R}$:

$$\psi(z) = \psi_0 + \frac{\sqrt{\pi}}{2} \left[\operatorname{erfi}(E_1(\lambda z)) - \operatorname{erfi}(E_1(\lambda z_0))\right] + i\Omega_\gamma. \tag{34}$$

Summary table

Operation	Result	Remark
$\partial_v \phi$	$2E_1 \cdot e^{E_1^2}$	generalized eigenequation
$\partial_v^2 \phi$	$(4E_1^2 + 2) \cdot e^{E_1^2}$	factors through ϕ
v-primitive	$\frac{\sqrt{\pi}}{2}\operatorname{erfi}(E_1)$	imaginary error function
Total v-integral	diverges	e^{w^2} not integrable on $\mathbb{R}^+$
v-ODE $\partial_v \psi = \phi$	solution via erfi	error function
Ω_γ $(\lambda = i\mu)$	$\pi/2 - \operatorname{Si}(\mu z_0)$	frozen, invisible to all

Remark 24 (Two functions, two special functions). *The two complete studies reveal a pattern: the v-primitive of a v-elementary function is a classical special function.*

$\phi(z)$	*v-primitive*	*Domain*
$\sin(E_1(\lambda z)^2)$	*Fresnel integral S*	*optics, diffraction*
$e^{E_1(\lambda z)^2}$	*error function* erfi	*probability, statistics*

Neither connection was anticipated. Both emerge from the composition $F \circ \Phi$ forced by the v-calculus structure. This suggests that the v-calculus provides a natural unifying framework for classical special functions.

10.5 A third complete study: $\phi(z) = \sec(E_1(\lambda z))$

We study $\phi(z) = \sec(E_1(\lambda z)) = 1/\cos(E_1(\lambda z))$, i.e., $F(w) = \sec(w)$. This is the first example with a *finite* radius of convergence and *transcendental singularities* — a phenomenon specific to the v-calculus.

v-Derivative

$F'(w) = \sec(w)\tan(w)$, so:

$$\partial_v \phi(z) = \sec(E_1(\lambda z)) \cdot \tan(E_1(\lambda z)) = \phi(z) \cdot \mathbf{T}_v(z), \tag{35}$$

where $\mathbf{T}_v(z) := \tan(E_1(\lambda z))$ is the v-tangent. This is a generalized eigenequation $\partial_v \phi = \phi \cdot \mathbf{T}_v$.

Radius of convergence

$F(w) = \sec(w)$ has singularities at $w = \pi/2 + k\pi$, $k \in \mathbb{Z}$. By Theorem 5:

$$R_v(\phi, z_0) = \operatorname{dist}\left(E_1(\lambda z_0),\, \left\{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\right\}\right). \tag{36}$$

This is the first example with finite R_v depending on z_0 — the v-Taylor series of ϕ converges only near z_0, not globally.

Transcendental singularities

The singularities of ϕ in the z-plane are the points z^* satisfying:

$$E_1(\lambda z^*) = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}. \tag{37}$$

These points have no closed-form expression in z — they are defined *implicitly* by the transcendental equation $E_1(\lambda z^*) = c$. This is a new species of singularity: not a pole of a rational function, not a branch point of a root, but a singularity localized by E_1.

Remark 25 (Transcendental singularities). *In classical analysis, singularities of* $\sec(x)$ *are at* $x = \pi/2 + k\pi$ *— explicit, regularly spaced on the real line. In the* v*-calculus, singularities of* $\sec(E_1(\lambda z))$ *are at the solutions of* $E_1(\lambda z^*) = \pi/2 + k\pi$ *— transcendental equations in* z *with no elementary solution. The geometry of singularities is transported and transformed by* $\Phi = E_1(\lambda z)$*.*

v-Primitive

$G'(w) = \sec(w)$ has the classical primitive $G(w) = \ln|\sec w + \tan w|$. Therefore:

$$\boxed{\Phi_f(z) = \ln\left|\sec(E_1(\lambda z)) + \tan(E_1(\lambda z))\right|.} \tag{38}$$

Unlike the Fresnel and error function cases, this primitive is not a named special function — it is an elementary combination of sec and tan under a logarithm, transported by Φ.

Definite v-integral

Along an integral curve from z_0 to z_1 (avoiding singularities):

$$\int_\gamma \phi \, d_v t = \ln\left|\sec(E_1(\lambda z_1)) + \tan(E_1(\lambda z_1))\right| - \ln\left|\sec(E_1(\lambda z_0)) + \tan(E_1(\lambda z_0))\right|.$$

The integral diverges as $z \to z^*$ (transcendental singularity) — the v-secant is not integrable across its singularities.

Summary table

Operation	Result	Remark
$\partial_v \phi$	$\phi \cdot \mathbf{T}_v$	generalized eigenequation
R_v	$\mathrm{dist}(E_1(\lambda z_0), \pi/2 + k\pi)$	first finite radius
Singularities	$E_1(\lambda z^*) = \pi/2 + k\pi$	transcendental in z
v-primitive	$\ln\left\|\sec(E_1) + \tan(E_1)\right\|$	elementary, not special
v-integral	diverges near z^*	transcendental obstruction
Ω_γ $(\lambda = i\mu)$	$\pi/2 - \mathrm{Si}(\mu z_0)$	frozen, invisible to all

Remark 26 (Three examples, three regimes). *The three complete studies classify three distinct regimes of the* v*-calculus:*

$\phi(z)$	R_v	*v-primitive*	*Regime*
$\sin(E_1^2)$	$+\infty$	*Fresnel S*	*entire, special function*
$e^{E_1^2}$	$+\infty$	erfi	*entire, divergent integral*
$\sec(E_1)$	*finite*	*elementary*	*transcendental singularities*

Each regime reveals a different face of the v-calculus: special functions from entire F, divergence from rapid growth, and transcendental singularities from poles of F transported by E_1.

10.6 Definite v-integrals

By the fundamental theorem (Theorem 1), $\int_\gamma \partial_v\phi \, d_v t = \mathrm{Re}(\phi(\gamma(T))) - \mathrm{Re}(\phi(\gamma(0)))$.

Example 8 (v-integral of $\mathbf{E}_v$). *Along any integral curve γ of v from z_0 to z_1:*

$$\begin{aligned}\int_\gamma \mathbf{E}_v \, d_v t &= \mathrm{Re}(\mathbf{E}_v(z_1)) - \mathrm{Re}(\mathbf{E}_v(z_0)) \\ &= e^{-\mathrm{Ci}(\lambda z_1)}\cos(\Omega_\gamma) - e^{-\mathrm{Ci}(\lambda z_0)}\cos(\Omega_\gamma) \\ &= \cos(\Omega_\gamma)\left(e^{-\mathrm{Ci}(\lambda z_1)} - e^{-\mathrm{Ci}(\lambda z_0)}\right),\end{aligned}$$

using $\mathrm{Re}(\mathbf{E}_v(z)) = e^{-\mathrm{Ci}(\lambda z)}$ and the constancy of Ω_γ along γ.

Example 9 (v-integral of $\mathbf{C}_v$). *Since the v-primitive of $\mathbf{C}_v$ is $\mathbf{S}_v$, and for $\lambda > 0$, $z > 0$ real, $\mathbf{S}_v(z) = \sin(E_1(\lambda z)) \in \mathbb{R}$:*

$$\int_\gamma \mathbf{C}_v \, d_v t = \mathrm{Re}(\mathbf{S}_v(z_1)) - \mathrm{Re}(\mathbf{S}_v(z_0)) = \sin(E_1(\lambda z_1)) - \sin(E_1(\lambda z_0)).$$

Exactly as $\int_{x_0}^{x_1} \cos x \, dx = \sin x_1 - \sin x_0$.

10.7 v-Differential equations

Example 10 (v-Bernoulli: $\partial_v\phi = \phi^2$). *With initial condition $\phi(z_0) = \phi_0 \in \mathbb{R}$.*

The reduction via $w = E_1(\lambda z)$ gives the real Bernoulli ODE $G'(w) = G(w)^2$, $G(w_0) = \phi_0$, with solution $G(w) = \phi_0/(1 - \phi_0(w - w_0))$. Recomposing:

$$\phi(z) = \frac{\phi_0}{1 - \phi_0\big(E_1(\lambda z) - E_1(\lambda z_0)\big)} + i\Omega_\gamma. \tag{39}$$

The solution blows up when $E_1(\lambda z) = E_1(\lambda z_0) + 1/\phi_0$, i.e., when the generating primitive reaches the critical value $w_0 + 1/\phi_0$.

Example 11 (v-linear: $\partial_v\phi = \mathbf{P}_v^1(z) = E_1(\lambda z)$). *With $\phi(z_0) = \phi_0 \in \mathbb{R}$.*

Reduction: $G'(w) = w$, $G(w_0) = \phi_0$, giving $G(w) = (w^2 - w_0^2)/2 + \phi_0$. Recomposing:

$$\phi(z) = \frac{E_1(\lambda z)^2 - E_1(\lambda z_0)^2}{2} + \phi_0 + i\Omega_\gamma = \frac{\mathbf{P}_v^2(z) - \mathbf{P}_v^2(z_0)}{2} + \phi_0 + i\Omega_\gamma.$$

Exactly as $\int w \, dw = w^2/2$, transported by Φ.

Example 12 (v-harmonic oscillator: $\partial_v^2\phi = -\phi$). *This is the v-ODE whose solutions are $\mathbf{S}_v$ and $\mathbf{C}_v$. General solution with initial conditions $\phi(z_0) = A \in \mathbb{R}$, $\partial_v\phi(z_0) = B \in \mathbb{R}$:*

$$\phi(z) = A\,\mathbf{C}_v(z) + B\,\mathbf{S}_v(z) + i\Omega_\gamma,$$

by direct analogy with $y'' = -y$ having solutions $A\cos x + B\sin x$. Verification: $\partial_v(A\mathbf{C}_v + B\mathbf{S}_v) = -A\mathbf{S}_v + B\mathbf{C}_v$, $\partial_v^2(A\mathbf{C}_v + B\mathbf{S}_v) = -A\mathbf{C}_v - B\mathbf{S}_v = -(A\mathbf{C}_v + B\mathbf{S}_v)$. ✓

10.8 Three structural observations from the examples

Observation 1: exact parallelism with real calculus. Every computation above is identical to its real counterpart, with x replaced by $E_1(\lambda z)$. The v-calculus computes exactly like the real calculus in the coordinate Φ.

Observation 2: Ω_γ appears systematically. In every v-ODE solution, Ω_γ appears as $+i\Omega_\gamma$, determined once by the initial condition $\phi(z_0)$ and never modified by any subsequent computation. It is a constant of integration of a purely complex nature, absent from the real calculus.

Observation 3: singularities come from E_1^{-1}. The blowup in Example 4 occurs when $E_1(\lambda z)$ reaches a critical value — not when z itself reaches a classical singularity. All analytic difficulties (branch cuts, blowup times, domains) are concentrated in the geometry of E_1 and its inverse, not in the algebraic structure of the v-calculus.

10.9 A complete worked example: derivative, integral, and Taylor series

We fix $v(z) = ze^z$ ($\lambda = 1$), $\Phi(z) = E_1(z)$, $\phi(z) = \mathbf{E}_v(z) = e^{E_1(z)}$, and $z_0 = 1$. Numerically: $E_1(1) \approx 0.2194$, $E_1(2) \approx 0.0489$.

Step 1 — v-Derivative at $z_0 = 1$.

$$\partial_v \mathbf{E}_v(z) = \frac{d}{dz} e^{E_1(z)} \cdot v(z) = e^{E_1(z)} \cdot \frac{-e^{-z}}{z} \cdot ze^z = e^{E_1(z)} = \mathbf{E}_v(z).$$

At $z_0 = 1$:

$$\partial_v \mathbf{E}_v(1) = e^{E_1(1)} \approx e^{0.2194} \approx 1.245.$$

The eigenequation $\partial_v \mathbf{E}_v = \mathbf{E}_v$ is confirmed: the v-derivative of $\mathbf{E}_v$ at any point equals $\mathbf{E}_v$ at that point, exactly as $(e^x)' = e^x$ in the real calculus.

Step 2 — Definite v-Integral from $z_0 = 1$ to $z_1 = 2$.

By the fundamental theorem (Theorem 1) and since $\partial_v \mathbf{E}_v = \mathbf{E}_v$, integrating $\mathbf{E}_v$ along the real segment $\gamma(t) = t$:

$$\int_1^2 \mathbf{E}_v \, d_v t = \mathrm{Re}(\mathbf{E}_v(2)) - \mathrm{Re}(\mathbf{E}_v(1)) = e^{E_1(2)} - e^{E_1(1)} \approx 1.050 - 1.245 = -0.195.$$

The result is negative because E_1 is decreasing: $\mathbf{E}_v$ diminishes as z moves away from the origin, consistently with $\mathbf{E}_v(z) \to 1$ as $z \to +\infty$.

Note: $\Omega_\gamma = \mathrm{Im}(e^{E_1(1)}) = 0$ since $z \in \mathbb{R}$ — the topological invariant vanishes on the real axis, as predicted by Proposition **??**.

Step 3 — v-Taylor Series at $z_0 = 1$.

Since $F(w) = e^w$ and $\partial_v^n \mathbf{E}_v = \mathbf{E}_v$ for all n, every coefficient equals $e^{E_1(1)} \approx 1.245$. The series is:

$$\mathbf{E}_v(z) = i\Omega_\gamma + e^{E_1(1)} \sum_{n=0}^{\infty} \frac{(E_1(z) - E_1(1))^n}{n!} = i\Omega_\gamma + e^{E_1(1)} \cdot e^{E_1(z) - E_1(1)} = i\Omega_\gamma + e^{E_1(z)}.$$

The series closes exactly to $\mathbf{E}_v(z)$ — confirming that the v-Taylor series of $\mathbf{E}_v$ is exact at every order, with infinite radius of convergence $R_v = +\infty$ (Corollary 5).

The three computations are summarized below.

Operation	Formula	Numerical value	Verification
v-Derivative $\partial_v \mathbf{E}_v(1)$	$e^{E_1(1)}$	≈ 1.245	$= \mathbf{E}_v(1)$ ✓
v-Integral on $[1,2]$	$e^{E_1(2)} - e^{E_1(1)}$	≈ -0.195	fundamental thm ✓
v-Taylor series	$i\Omega_\gamma + e^{E_1(z)}$	exact	$R_v = +\infty$ ✓

10.10 The v-calculus as a linearization tool for integration

A fundamental application of the v-calculus is the computation of integrals that are intractable in the classical setting. The key mechanism is the change of variable $w = E_1(\lambda z)$, which linearizes the integration along curves integral to v.

Theorem 12 (Integration via v-linearization). *Let $v(z) = ze^{\lambda z}$, $\Phi(z) = E_1(\lambda z)$, and let $g : \Phi(U) \to \mathbb{R}$ be a continuous function admitting a classical primitive G (i.e., $G' = g$). Let γ be an integral curve of v from z_0 to z_1. Then:*

$$\int_\gamma \frac{g(E_1(\lambda z))}{ze^{\lambda z}}\, dz = G(E_1(\lambda z_1)) - G(E_1(\lambda z_0)). \tag{40}$$

In particular, the left-hand side — classically intractable whenever g involves transcendental functions of E_1 — reduces to a simple evaluation of G at the endpoints.

Proof. Along γ, $\dot\gamma(t) = v(\gamma(t)) = ze^{\lambda z}$, so $dz = ze^{\lambda z}\, dt$. Substituting:

$$\int_\gamma \frac{g(E_1(\lambda z))}{ze^{\lambda z}}\, dz = \int_0^T g(E_1(\lambda z(t)))\, dt = \int_\gamma g(E_1(\lambda z))\, d_v t.$$

Since $\partial_v (G \circ \Phi)(z) = G'(E_1(\lambda z)) = g(E_1(\lambda z))$, the fundamental theorem (Theorem 1) gives:

$$\int_\gamma g(E_1(\lambda z))\, d_v t = \mathrm{Re}(G(E_1(\lambda z_1))) - \mathrm{Re}(G(E_1(\lambda z_0))) = G(E_1(\lambda z_1)) - G(E_1(\lambda z_0)),$$

since G is real-valued on $\Phi(U) \subset \mathbb{R}$. □

Remark 27 (Scope of the theorem). *The classical difficulty of $\int g(E_1(\lambda z))/(ze^{\lambda z})\, dz$ comes from the transcendental nature of E_1 — no change of variable in $\mathbb{R}$ resolves it. The v-calculus resolves it by recognizing that $1/(ze^{\lambda z}) = 1/v(z)$ is the differential of Φ: the integral is classically hard precisely because the integrand encodes the geometry of the v-field, which the v-calculus reads directly.*

Corollary 9 (Canonical integrable family). *The following integrals, classically without elementary closed form, are computed exactly by the v-calculus:*

$g(w)$	*Integral*	*Result*
$\sin w$	$\int_\gamma \frac{\sin(E_1(\lambda z))}{ze^{\lambda z}}\, dz$	$-\cos(E_1(\lambda z_1)) + \cos(E_1(\lambda z_0))$
e^w	$\int_\gamma \frac{e^{E_1(\lambda z)}}{ze^{\lambda z}}\, dz$	$e^{E_1(\lambda z_1)} - e^{E_1(\lambda z_0)}$
w^n	$\int_\gamma \frac{E_1(\lambda z)^n}{ze^{\lambda z}}\, dz$	$\dfrac{E_1(\lambda z_1)^{n+1} - E_1(\lambda z_0)^{n+1}}{n+1}$
$\sin(w^2)$	$\int_\gamma \frac{\sin(E_1(\lambda z)^2)}{ze^{\lambda z}}\, dz$	$\sqrt{\frac{\pi}{2}}\left[S(\cdot\, E_1(z_1)) - S(\cdot\, E_1(z_0))\right]$

Example 13 (A classically intractable integral). *We compute:*

$$I = \int_\gamma \frac{\sin(E_1(z))}{ze^z}\, dz,$$

where $\lambda = 1$ and γ is the integral curve of $v(z) = ze^z$ from $z_0 = 1$ to $z_1 = 2$.

Classical approach. *The function $\sin(E_1(z))/(ze^z)$ has no elementary primitive in z — no computer algebra system can compute this integral in closed form.*

v-calculus approach. *Along γ, $dz = ze^z\, dt$, so:*

$$I = \int_\gamma \sin(E_1(z))\, d_v t = \int_\gamma \mathbf{S}_v\, d_v t.$$

Since the v-primitive of $\mathbf{S}_v$ is $-\mathbf{C}_v$ (i.e., $\partial_v(-\mathbf{C}_v) = \mathbf{S}_v$), the fundamental theorem gives:

$$I = -\cos(E_1(z_1)) + \cos(E_1(z_0)) = -\cos(E_1(2)) + \cos(E_1(1)).$$

Numerical evaluation. *$E_1(1) \approx 0.2194$, $E_1(2) \approx 0.0489$:*

$$I = -\cos(0.0489) + \cos(0.2194) \approx -0.9988 + 0.9761 = -0.0227.$$

Note: $\Omega_\gamma = 0$ since $z \in \mathbb{R}$ — the topological invariant is invisible on the real axis, as expected.

Summary. *A classically intractable integral is computed in two lines via the v-calculus linearization, yielding an exact closed form and a numerical value.*

11 v-Fourier Analysis

11.1 The v-trigonometric basis

The v-trigonometric functions of frequency $n \geq 1$ are:

$$\mathbf{S}_{v,n}(z) := \sin(n\, E_1(\lambda z)), \qquad \mathbf{C}_{v,n}(z) := \cos(n\, E_1(\lambda z)). \tag{41}$$

For $n = 1$: $\mathbf{S}_{v,1} = \mathbf{S}_v$ and $\mathbf{C}_{v,1} = \mathbf{C}_v$ of Section 9. Their v-derivatives satisfy:

$$\partial_v \mathbf{S}_{v,n} = n\mathbf{C}_{v,n}, \qquad \partial_v \mathbf{C}_{v,n} = -n\mathbf{S}_{v,n}, \qquad \mathbf{S}_{v,n}^2 + \mathbf{C}_{v,n}^2 = 1.$$

Each pair $(\mathbf{S}_{v,n}, \mathbf{C}_{v,n})$ solves the v-harmonic oscillator $\partial_v^2 \phi = -n^2\phi$ of frequency n.

11.2 Orthogonality

The v-integral along a curve γ such that $\Phi(\gamma(T)) - \Phi(\gamma(0)) = 2\pi$ is called a *v-period integral* and denoted $\int_{\gamma_{2\pi}}$. In the $w = \Phi(z)$ coordinate, $\int_{\gamma_{2\pi}} f\, d_v t = \int_{w_0}^{w_0+2\pi} F(w)\, dw$.

Theorem 13 (v-orthogonality). *For all $m, n \geq 1$:*

$$\int_{\gamma_{2\pi}} \mathbf{S}_{v,m}(z)\, \mathbf{S}_{v,n}(z)\, d_v t = \begin{cases} \pi & m = n \\ 0 & m \neq n, \end{cases} \tag{42}$$

$$\int_{\gamma_{2\pi}} \mathbf{C}_{v,m}(z)\, \mathbf{C}_{v,n}(z)\, d_v t = \begin{cases} \pi & m = n \\ 0 & m \neq n, \end{cases} \tag{43}$$

$$\int_{\gamma_{2\pi}} \mathbf{S}_{v,m}(z)\, \mathbf{C}_{v,n}(z)\, d_v t = 0 \quad \textit{for all } m, n. \tag{44}$$

Proof. In the w-coordinate, each identity reduces to the classical orthogonality of $\{\sin(nw), \cos(nw)\}$ on $[w_0, w_0 + 2\pi]$, which holds by direct integration. □

11.3 The v-Fourier series

Definition 18 (v-Fourier coefficients). *Let $\phi \in \mathcal{D}_v(U)$ with $\phi = F \circ \Phi$. The* v-Fourier coefficients *of ϕ on a v-period are:*

$$a_n := \frac{1}{\pi} \int_{\gamma_{2\pi}} \phi \, \mathbf{C}_{v,n} \, d_v t = \frac{1}{\pi} \int_{w_0}^{w_0+2\pi} F(w) \cos(nw) \, dw, \quad b_n := \frac{1}{\pi} \int_{\gamma_{2\pi}} \phi \, \mathbf{S}_{v,n} \, d_v t. \tag{45}$$

All coefficients $a_n, b_n \in \mathbb{R}$.

Theorem 14 (v-Fourier series). *If $F = \phi \circ \Phi^{-1}$ is square-integrable on $[w_0, w_0 + 2\pi]$, then ϕ admits the v-Fourier expansion:*

$$\phi(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \, \mathbf{C}_{v,n}(z) + b_n \, \mathbf{S}_{v,n}(z) \big] + i\Omega_\gamma, \tag{46}$$

convergent in the v-L^2 norm $\|\phi\|_{v,2}^2 := \int_{\gamma_{2\pi}} |\mathrm{Re}(\phi)|^2 \, d_v t$.

Proof. In the w-coordinate, this is the classical Fourier series of F on $[w_0, w_0 + 2\pi]$, transported back to z-coordinates via Φ. The L^2 convergence is the classical L^2 convergence of Fourier series, and the additive term $i\Omega_\gamma$ is a constant carried through unchanged. □

11.4 v-Parseval identity

Theorem 15 (v-Parseval). *Under the hypotheses of Theorem 14:*

$$\frac{1}{\pi} \int_{\gamma_{2\pi}} |\mathrm{Re}(\phi)|^2 \, d_v t = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2). \tag{47}$$

The spectral energy equals the v-L^2 norm of the dynamic part only.

Proof. Classical Parseval identity applied to F in the w-coordinate. □

11.5 The v-Fourier transform

For functions defined on all of $\Phi^{-1}(\mathbb{R})$, the *v-Fourier transform* of $\phi = F \circ \Phi$ is:

$$\hat{\phi}_v(\xi) := \int_{\gamma} \phi(z) \, e^{-i\xi\Phi(z)} \, d_v t = \int_{-\infty}^{+\infty} F(w) \, e^{-i\xi w} \, dw = \hat{F}(\xi). \tag{48}$$

It inherits all properties of the classical Fourier transform, transported by Φ. In particular:

$$\widehat{\partial_v \phi_v}(\xi) = i\xi \, \hat{\phi}_v(\xi), \qquad \widehat{\phi \star_v \psi}_v(\xi) = \hat{\phi}_v(\xi) \cdot \hat{\psi}_v(\xi). \tag{49}$$

11.6 Spectral invisibility of Ω_γ

Theorem 16 (Spectral invisibility). *The topological invariant* $\Omega_\gamma = \mathrm{Im}(\phi(z_0))$ *does not appear in any v-Fourier coefficient:*

$$\frac{\partial a_n}{\partial \Omega_\gamma} = 0, \qquad \frac{\partial b_n}{\partial \Omega_\gamma} = 0 \quad \textit{for all } n \geq 0.$$

The v-Fourier series (46) *reconstructs the dynamic part* $\mathrm{Re}(\phi)$ *exactly, but cannot recover* Ω_γ *from any finite or infinite collection of spectral coefficients.*

Proof. From (45), $a_n = \frac{1}{\pi}\int F(w)\cos(nw)\,dw$ and $F(w) = \mathrm{Re}(\phi(\Phi^{-1}(w)))$. Since $\Omega_\gamma = \mathrm{Im}(\phi)$ is a real constant additive to the imaginary part of ϕ only, it does not affect $F = \mathrm{Re}(\phi \circ \Phi^{-1})$. Hence a_n and b_n are independent of Ω_γ. □

Remark 28 (Fourier manifestation of maximal bifurcation). *Theorem 16 is the Fourier-analytic version of the maximal bifurcation theorem (Theorem 8): just as* Ω_γ *is invisible to differentiation, integration, Taylor expansion, and differential equations, it is now shown to be invisible to the entire Fourier spectrum. No spectral decomposition — finite or infinite — can detect or reconstruct* Ω_γ*. It lies strictly outside the reach of harmonic analysis.*

The v-Fourier expansion (46) *therefore has an irreducible structure:*

$$\phi = \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \mathbf{C}_{v,n} + b_n \mathbf{S}_{v,n})}_{\textit{Fourier spectrum (dynamic part)}} + \underbrace{i\Omega_\gamma}_{\textit{outside spectrum (topological part)}}.$$

11.7 Numerical verification

We verify all results numerically at $\lambda = 1$, $w_0 = 0.5$, working in the w-coordinate on the interval $[0.5,\ 0.5 + 2\pi]$.

Orthogonality

Integral	Numerical value	Expected
$\int \mathbf{S}_{v,1} \cdot \mathbf{S}_{v,2}\, d_v t$	0.00×10^{0}	0
$\int \mathbf{S}_{v,1} \cdot \mathbf{S}_{v,3}\, d_v t$	2.18×10^{-17}	0
$\int \mathbf{S}_{v,2} \cdot \mathbf{S}_{v,3}\, d_v t$	-2.78×10^{-16}	0
$\int \mathbf{S}_{v,1} \cdot \mathbf{S}_{v,1}\, d_v t$	3.1415926536	π
$\int \mathbf{S}_{v,2} \cdot \mathbf{S}_{v,2}\, d_v t$	3.1415926536	π
$\int \mathbf{C}_{v,1} \cdot \mathbf{C}_{v,2}\, d_v t$	-1.11×10^{-16}	0
$\int \mathbf{C}_{v,1} \cdot \mathbf{C}_{v,1}\, d_v t$	3.1415926536	π
$\int \mathbf{S}_{v,1} \cdot \mathbf{C}_{v,1}\, d_v t$	-2.18×10^{-16}	0

Maximum error: 2.78×10^{-16} (machine precision). ✓

Fourier series convergence

For $\phi = \mathbf{P}_v^2$ ($F(w) = w^2$), reconstruction at $w = w_0 + \pi$ with $N = 50$ terms: exact value = 13.2612, reconstructed = 13.2604, error = 8.16×10^{-4}. Convergence is slow for w^2 (typical for functions with endpoint discontinuities of derivatives) but confirmed.

Parseval identity

For $F(w) = \cos(w)$: $(1/\pi)\int \cos^2(w)\,dw = 1.0000000000$, $a_0^2/2+\sum(a_n^2+b_n^2) = 1.0000000000$, error $= 0.00 \times 10^0$. ✓

Spectral invisibility of Ω_γ

For $\phi = w^2 + i\Omega_\gamma$, the coefficient a_1 for four distinct values of Ω_γ:

Ω_γ	a_1
0.0000	10.4938203249
0.5000	10.4938203249
$\pi/4$	10.4938203249
1.2340	10.4938203249

All values identical. Error $= 0.00 \times 10^0$. ✓

12 Summary

We collect the main results of this working paper.

R1. Algebraic structure (Section 2). $\mathcal{D}_v(U)$ is a real vector space but not a $\mathbb{C}$-module. The Leibniz rule holds under condition (1); real-valued functions form a stable subring. The chain rule holds when the outer function has real derivative.

R2. Fundamental theorem (Theorem 1). $\int_\gamma \partial_v\phi\, d_v t = \mathrm{Re}(\phi(\gamma(T))) - \mathrm{Re}(\phi(\gamma(0)))$: integration recovers only the real part.

R3. Fundamental decomposition (Corollary 2). Every $\phi \in \mathcal{D}_v(U)$ splits into a dynamic part (integrated by ∂_v) and a frozen topological part Ω_γ.

R4. Integrability and completeness (Definitions 5, 6; Propositions 7, 9). Integrable fields admit rich calculi generated by Φ. Complete fields give a calculus isomorphic to the real calculus via $w = \Phi(z)$.

R5. Residues generate topology (Corollary 3). Non-zero residues of $1/v$ are the unique source of non-trivial topological invariants. Three levels of the v-calculus are separated by the residue structure of v.

R6. Iterated v-derivatives (Theorem 2). $\partial_v^n\phi(z) = F^{(n)}(\Phi(z))$: iteration diagonalizes completely in the Φ-coordinate.

R7. v-Taylor series (Theorem 3). The v-Taylor series has real coefficients $\partial_v^n\phi(z_0)/n! \in \mathbb{R}$ and converges in the Φ-coordinate.

R8. Topological invariant is Taylor-invisible (Theorem 4). Ω_γ is absent from every coefficient of the v-Taylor series and irrecoverable from any collection of v-derivatives.

R9. Reduction theorem (Theorem 6). Every v-ODE $\partial_v\phi = F(\phi, z)$ reduces via $w = \Phi(z)$ to a real first-order ODE, with Ω_γ as a free parameter. The classification of v-ODEs is isomorphic to that of real first-order ODEs (Corollary 6).

R10. Canonical classes of v-ODEs (Propositions 11–13). Constant, linear, and separable v-ODEs are solved explicitly, recovering the $\mathcal{F}_\lambda$ linearization as the canonical constant case.

R11. Unification of the A_n hierarchy (Theorem 7). The entire A_n hierarchy is the family of solutions of the single equation $\partial_{v_n}\phi = -1$, for varying fields $v_n(z) = z^n e^{\lambda z}$. The hierarchy is unified under one equation type.

R12. Maximal bifurcation theorem (Theorem 8). The topological invariant Ω_γ is invisible to every operation of the v-calculus: differentiation, integration, Taylor expansion, and differential equations. It is determined once and for all by the initial condition.

R13. v-elementary functions (Section 9). The principle $\phi = F \circ \Phi$ with F' real-valued generates a complete family of v-elementary functions: the v-exponential $\mathbf{E}_v = e^{E_1(\lambda z)}$ (eigenfunction $\partial_v \mathbf{E}_v = \mathbf{E}_v$, exact propagator for $\partial_v \phi = \phi$); the v-trigonometric functions $\mathbf{S}_v$, $\mathbf{C}_v$ (satisfying $\partial_v \mathbf{S}_v = \mathbf{C}_v$, $\partial_v \mathbf{C}_v = -\mathbf{S}_v$, and the Pythagorean identity, and solving the v-harmonic oscillator $\partial_v^2 \phi = -\phi$); the v-powers $\mathbf{P}_v^n$ (natural basis for v-Taylor series); and the v-logarithm. The v-exponential carries the topological invariant Ω_γ in its phase, decomposing as amplitude × phase with the phase frozen and the amplitude evolved by ∂_v. The entire A_n hierarchy produces a coherent tower of v_n-exponentials $\mathbf{E}_{v_n} = e^{\lambda^{n-1} E_n(\lambda z)}$.

R14. v-Euler identity (Theorem 11; Propositions 23, 24). The two natural v-exponential objects $\mathbf{E}_v = e^{E_1(\lambda z)}$ and $\mathbf{C}_v + i\mathbf{S}_v = e^{iE_1(\lambda z)}$ are related by the exact identity $\mathbf{E}_v(z) = e^{(1-i)E_1(\lambda z)} \cdot (\mathbf{C}_v(z) + i\mathbf{S}_v(z))$. The corrector $e^{(1-i)E_1(\lambda z)}$ encodes an *amplitude-phase duality*: $\mathrm{Ci}(\mu z)$ and Ω_γ play exchanged roles in $\mathbf{E}_v$ and $\mathbf{C}_v + i\mathbf{S}_v$ — what is amplitude in one is phase in the other. The two objects coincide if and only if $E_1(\lambda z) \in \pi(1+i)\mathbb{Z}$.

R15. Extension to $\mathbb{C}^n$ (Section 8). The v-calculus extends to any dimension n with $\omega_v = \sum v_k \, dz_k$ replacing dz/v and normalization $\omega_v(v) = 1$. The *dimensional reduction theorem* (Theorem 9) shows that the v-calculus always reduces to dimension 1 via the scalar generating primitive $\Phi : U \to \mathbb{C}$. All results hold in $\mathbb{C}^n$ with identical proofs. The genuine n-dimensional novelty is the topology of the fibres $\mathcal{F}_c = \{\Phi(z) = c\}$ — complex hypersurfaces encoding information invisible to the one-dimensional reduction.

R16. Concrete calculations (Section 10). Five families of explicit calculations confirm the theory: v-derivatives (including the scaled eigenfunction $\partial_v(e^{2E_1}) = 2e^{2E_1}$), v-primitives ($\mathbf{E}_v$ is its own primitive; $\mathbf{S}_v$ is the primitive of $\mathbf{C}_v$), definite v-integrals (via the fundamental theorem), and v-ODEs (Bernoulli $\partial_v \phi = \phi^2$; linear $\partial_v \phi = \mathbf{P}_v^1$; harmonic oscillator $\partial_v^2 \phi = -\phi$ with general solution $A\mathbf{C}_v + B\mathbf{S}_v + i\Omega_\gamma$). Every computation is identical to its real analogue with x replaced by $E_1(\lambda z)$.

R17. v-Fourier analysis (Section 11; Theorems 13–16). The v-trigonometric functions $\{\mathbf{S}_{v,n}, \mathbf{C}_{v,n}\}$ of frequency n form an orthogonal basis for the v-L^2 space, with $\int_{\gamma_{2\pi}} \mathbf{S}_{v,m} \mathbf{S}_{v,n} \, d_v t = \pi \delta_{mn}$ (verified to 2.78×10^{-16}). Every v-differentiable function admits a v-Fourier series with real coefficients $a_n, b_n \in \mathbb{R}$, and the v-Parseval identity holds. The *spectral invisibility theorem* shows that Ω_γ is absent from every Fourier

coefficient — it lies strictly outside the Fourier spectrum for any value of Ω_γ (verified exactly). This is the Fourier-analytic manifestation of the maximal bifurcation theorem: no spectral decomposition can recover the topological invariant.

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References

[1] J. Brindel, *The $\mathcal{F}_\lambda$ Class: Algebraic Stability, Explicit Linearization, and Topological Invariants of a Canonical Family of ODEs*, Zenodo (2026). doi:10.5281/zenodo.19234507

[2] L. V. Ahlfors, *Complex Analysis*, 3rd ed., McGraw-Hill, New York (1979).

[3] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York (1987).